Origami and Constructible Numbers

(and some other stuff)

Tom Hull, Merrimack College
thull@merrimack.edu

Straightedge and Compass basic operations

- Given two points P_1 and P_2, we can draw the line P_1P_2.
- Given a point P and a line segment of length r, we can draw a circle centered at P with radius r.
- We can locate intersection points, if they exist, between lines and circles.

What are the Basic Operations of Origami?

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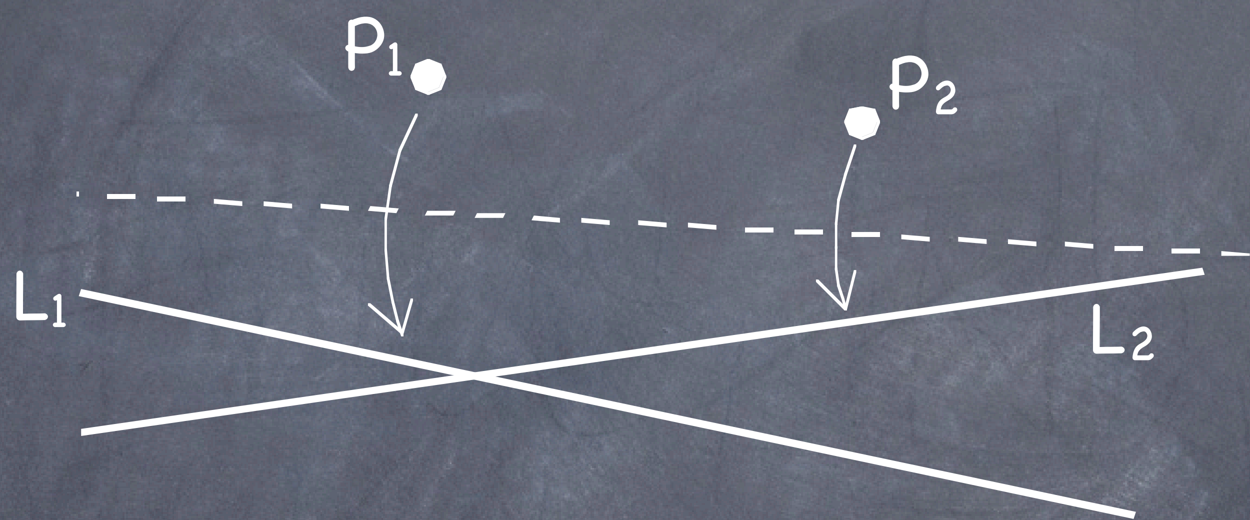
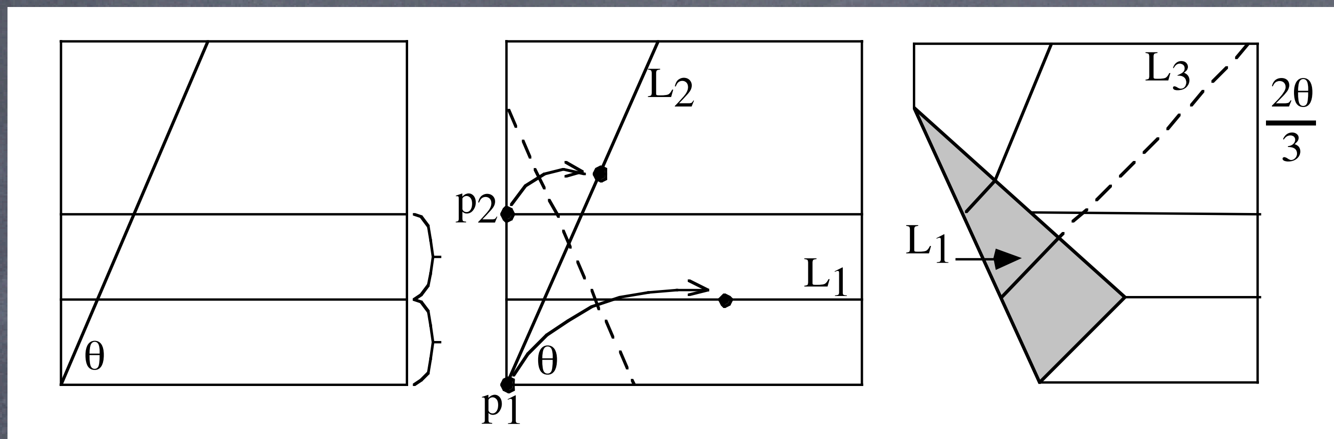
- Given two points P_1 and P_2, we can fold the crease line P_1P_2.
- Given two points P_1 and P_2, we can make a crease that puts P_1 onto P_2.
- Given two lines L_1 and L_2, we can make a crease that puts L_1 onto L_2.
- and so on.

The craziest BOO



The most important move in origami (probably)

Origami angle trisection



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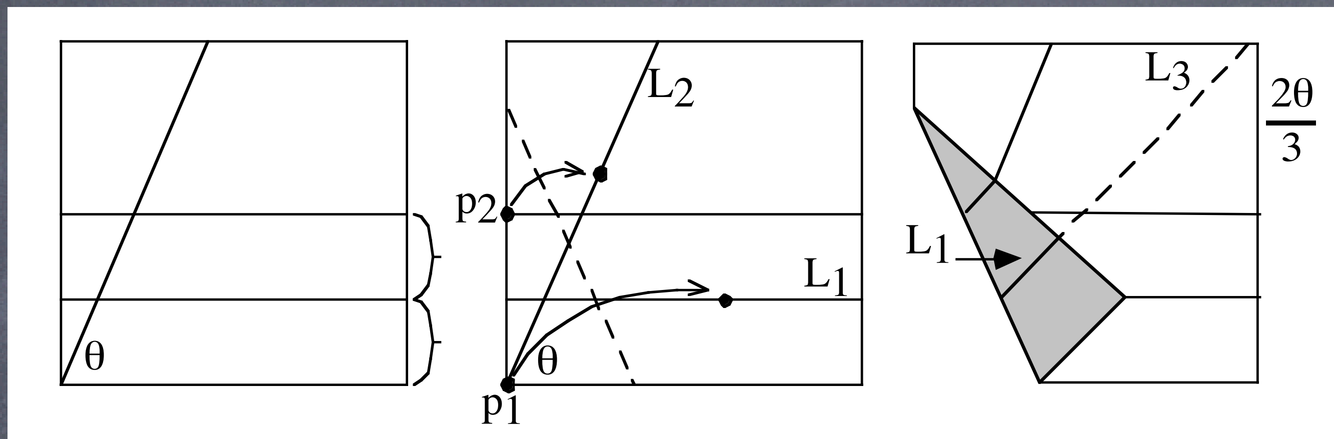
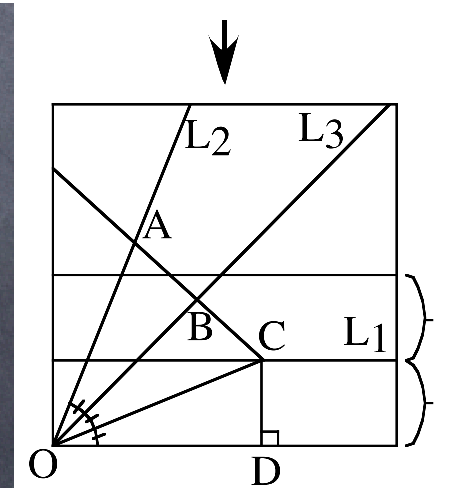
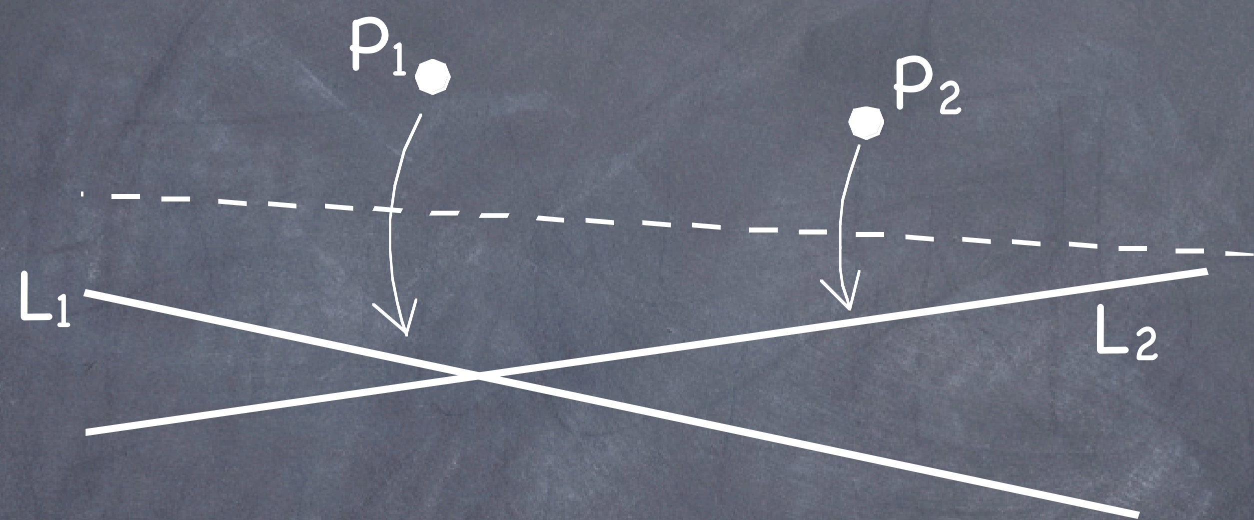


Proof:



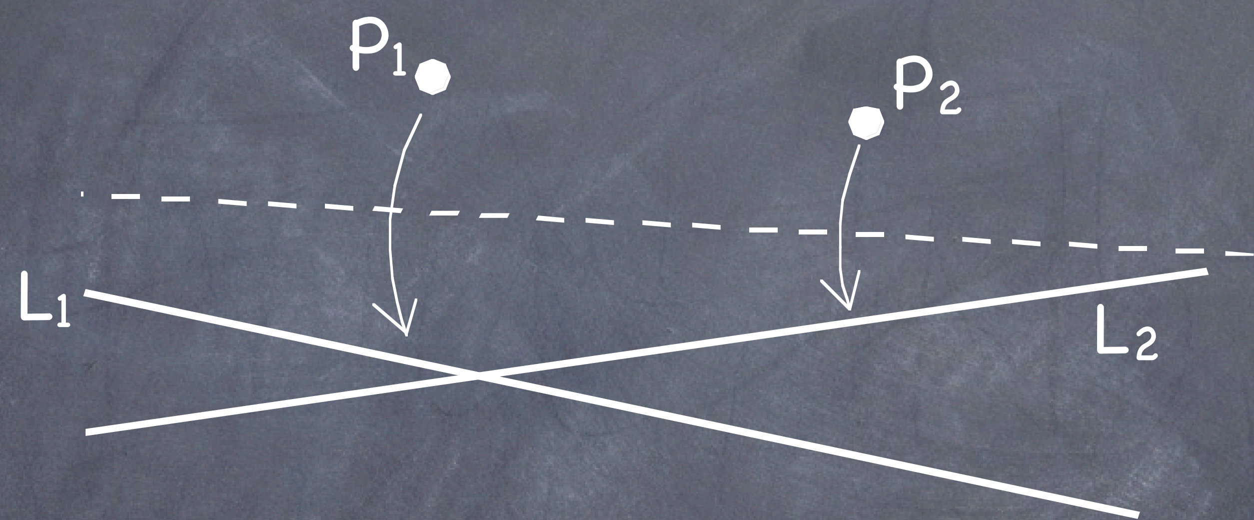
credit: Hisashi Abe, 1980

Origami angle trisection



The deal is: this origami move is actually solving a cubic equation.

Origami angle trisection

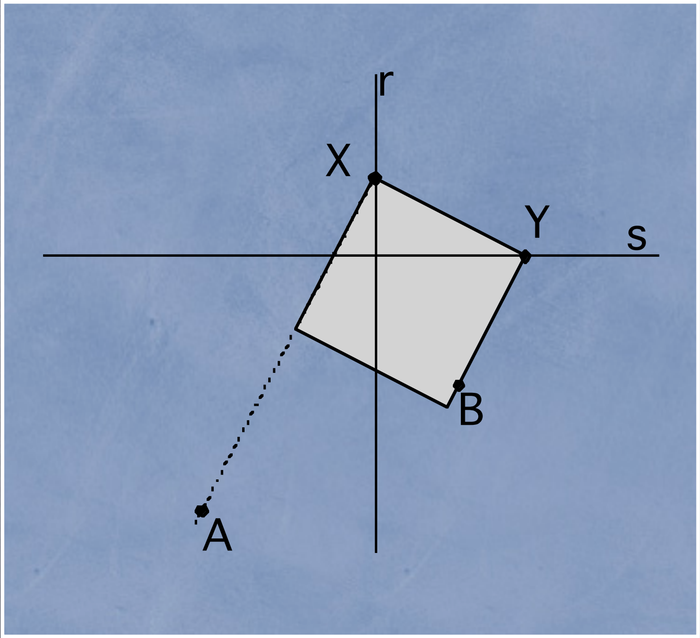


The deal is: this origami move is actually solving a cubic equation.
(Finding a simultaneous tangent to two parabolas.)

Origami can solve any cubic equation

Italian mathematician Margherita Beloch proved this in the 1930s. Here's her proof:

Consider the following construction problem:
Let A and B be two points and r and s two lines.
We want to construct a square that has A and B on opposite sides (or extensions) and has two adjacent vertices lying on the lines r and s.



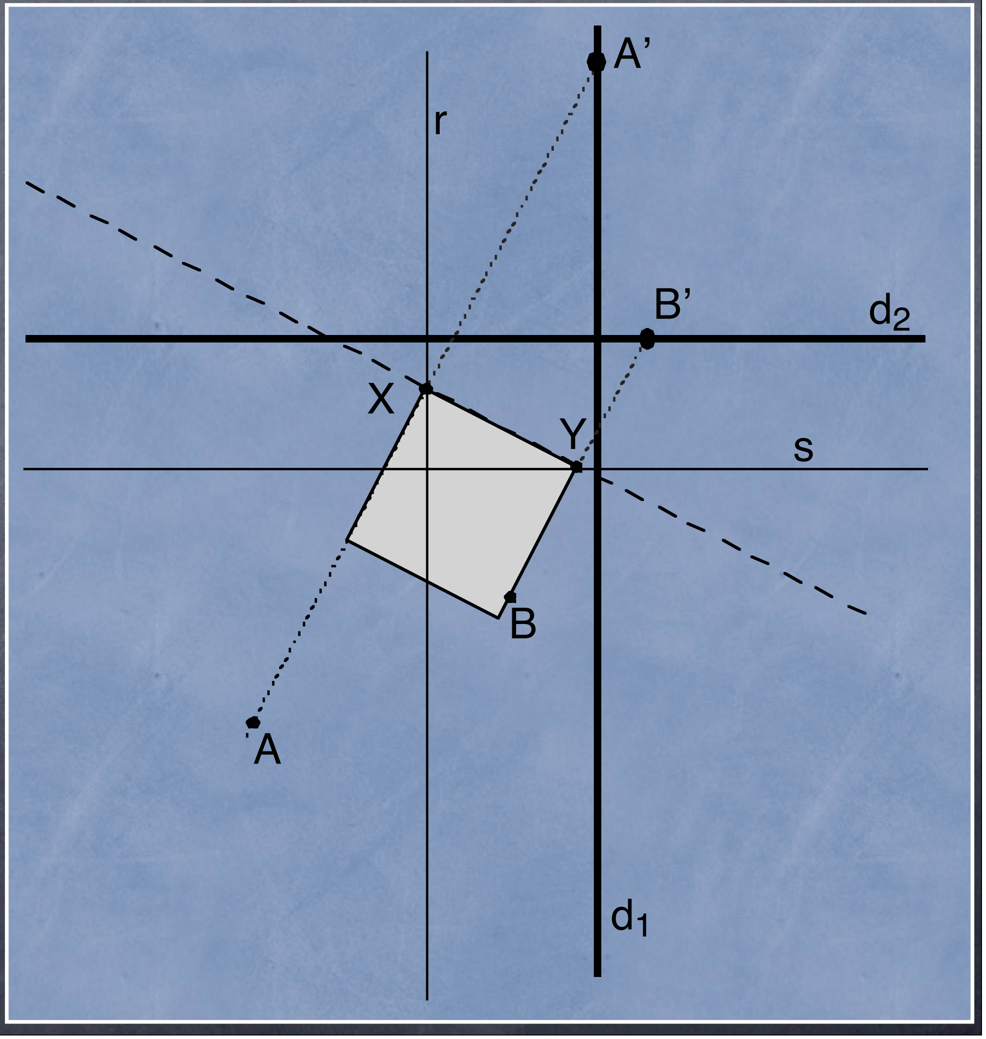
Origami can solve any cubic equation

We can make this square construction with origami.

Let d_1 be a line parallel to r, where dist(A, r) = dist(r, d_1).

Let d_2 be a line parallel to s, where dist(B, s) = dist(s, d_2).

Then fold A -> d_1 and B -> d_2 simultaneously. The crease gives the top of the square (XY).



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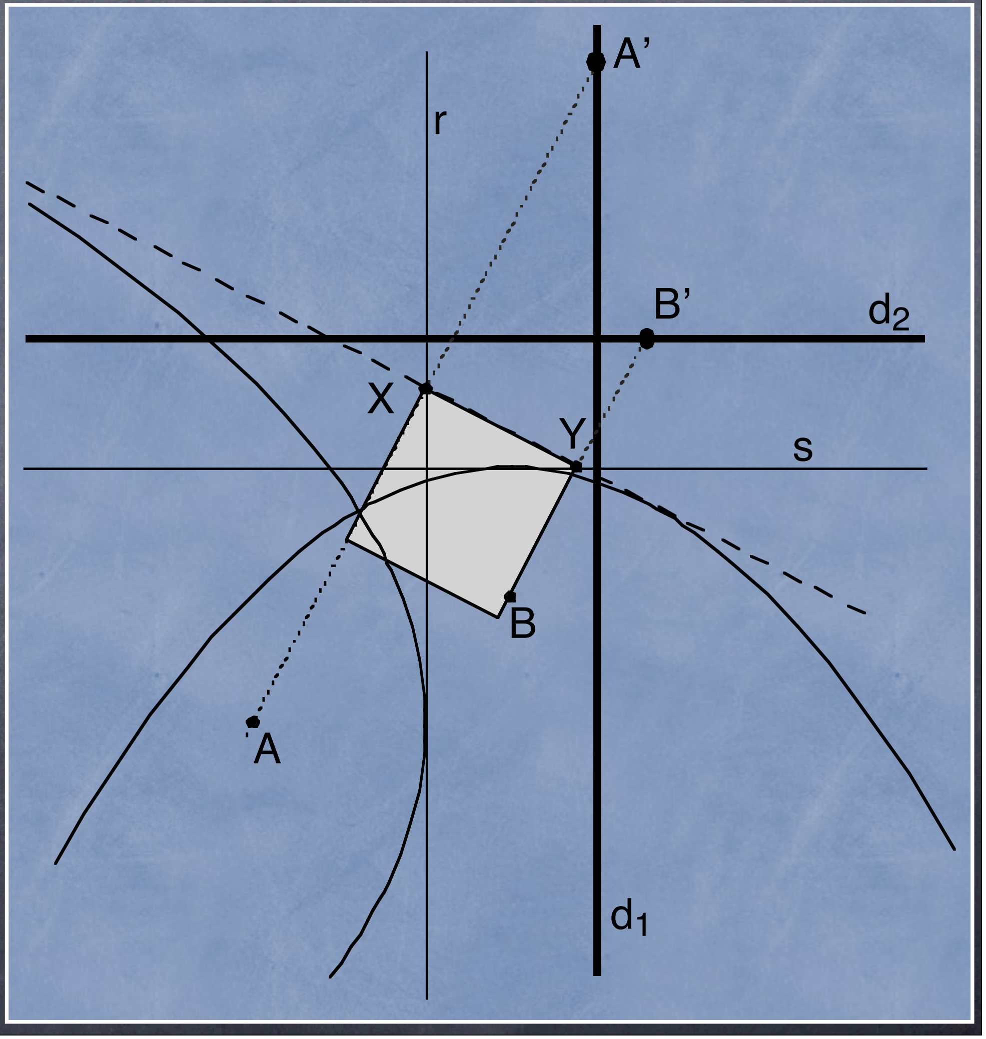
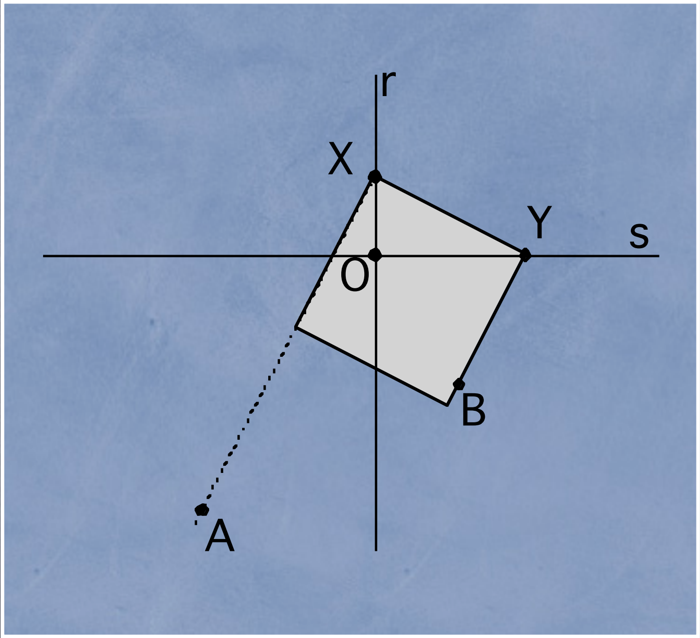


Origami can solve any cubic equation

Why is this like solving a cubic equation?
Beloch realized that this is just an application of Lill's method for finding real roots of a polynomial!

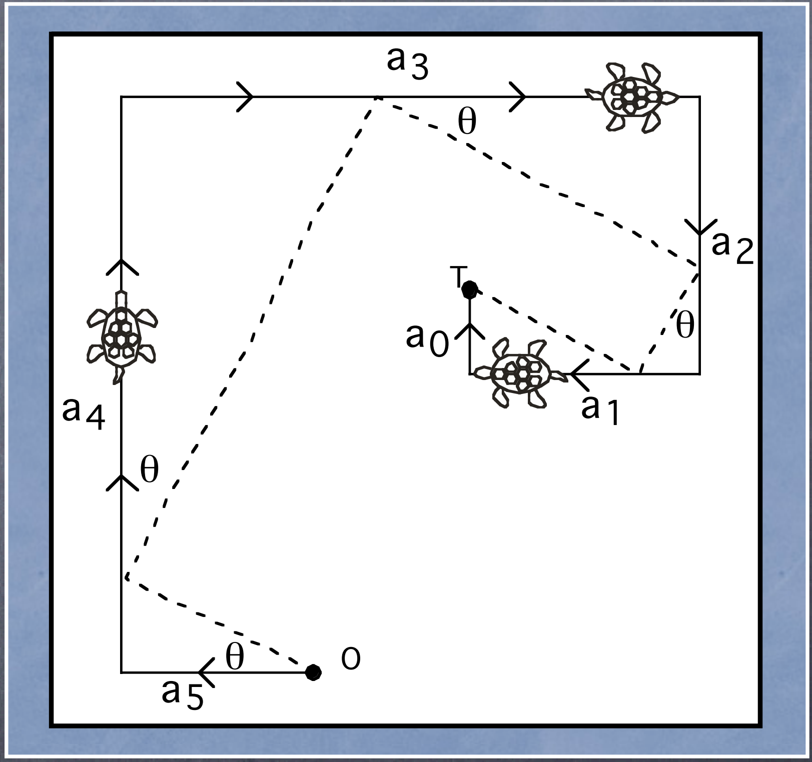


Origami can solve any cubic equation

Lill's geometric method for finding real roots of any polynomial:

$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 = 0$

Start at O, go a_n, turn 90°, go a_{n-1}, turn 90°, etc, ending at T.



Then shoot from O with an angle θ, bouncing off the walls at right angles, to hit T.

Then $x = -\tan\theta$ is a root.

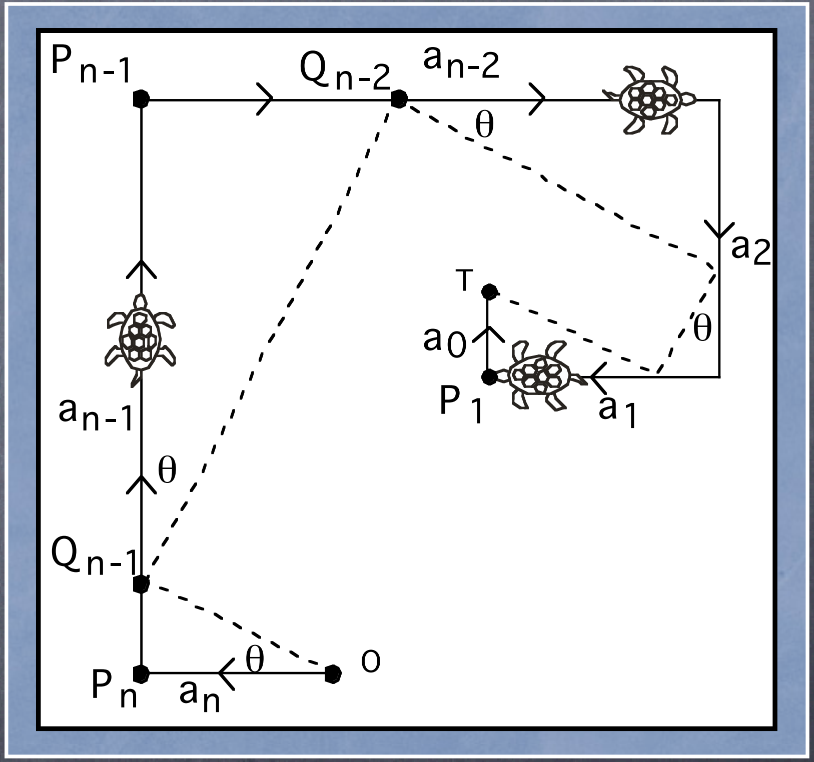
(Lill, 1867)

Origami can solve any cubic equation

Why does Lill's method work?

$P_nQ_{n-1} / a_n = \tan\theta = -x$

So $P_nQ_{n-1} = -a_nx$



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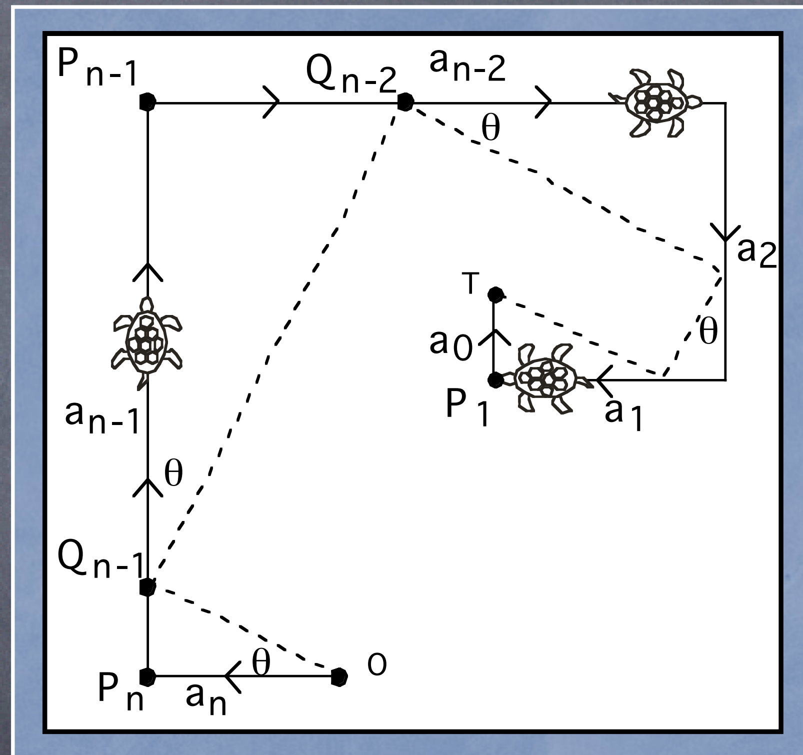
$P_{n-1}Q_{n-2} / (a_{n-1} - P_nQ_{n-1}) = -x$

So $P_{n-1}Q_{n-2} = -x(a_{n-1} + a_nx)$



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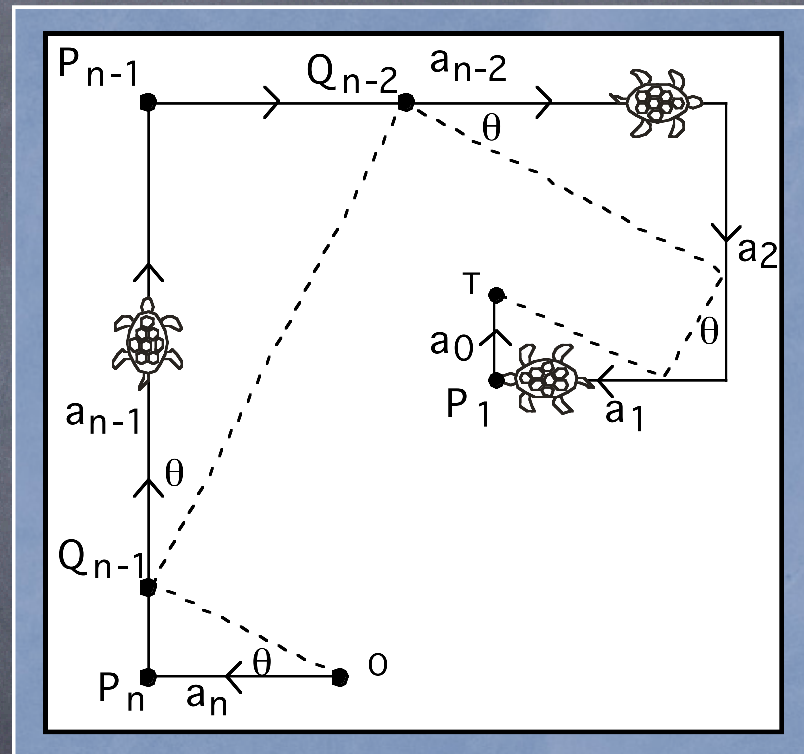

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So $P_{n-1} Q_{n-2} = -x(a_{n-1} + a_n x)$

Similarly,

$P_{n-2} Q_{n-3} = -x(a_{n-2} + x(a_{n-1} + a_n x))$

Continuing...

$a_0 = P_1 T = - a_1 x - a_2 x^2 - \ldots - a_{n-1} x^{n-1} - a_n x^n$

or, $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 = 0.$

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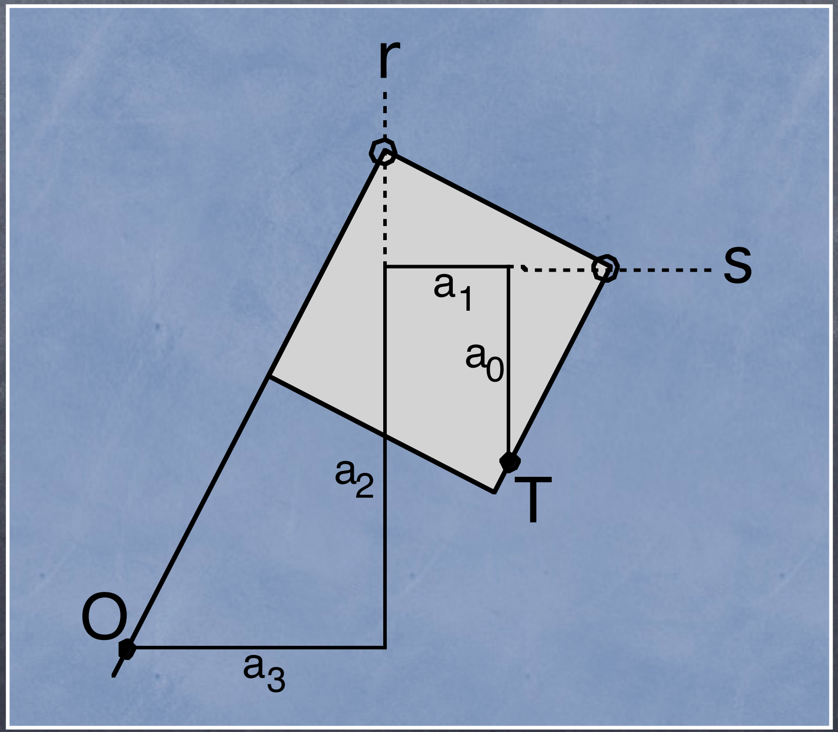
Why is this like solving a cubic equation?

Finding our "construction square" is the same as "shooting the turtle" in the n=3 case!



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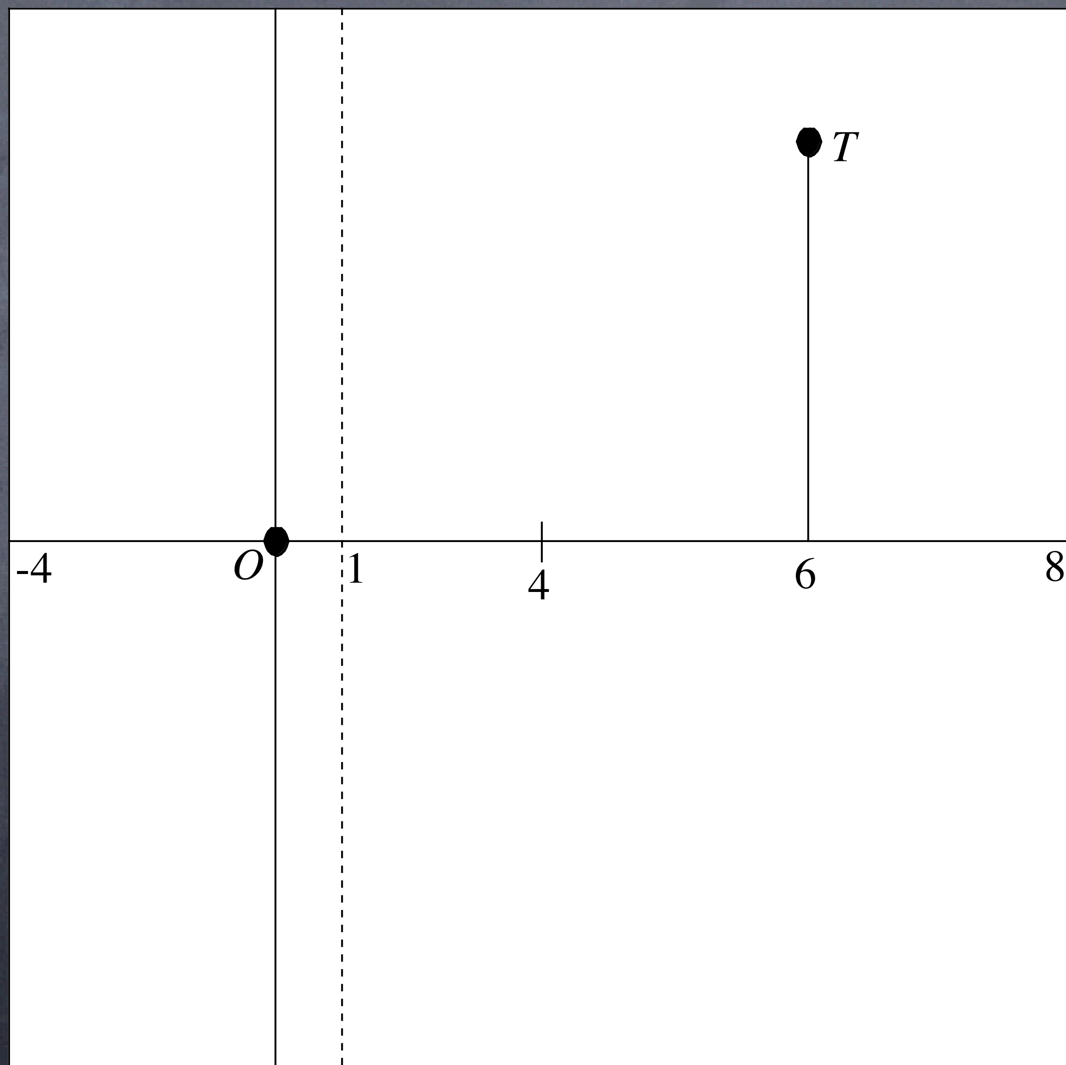

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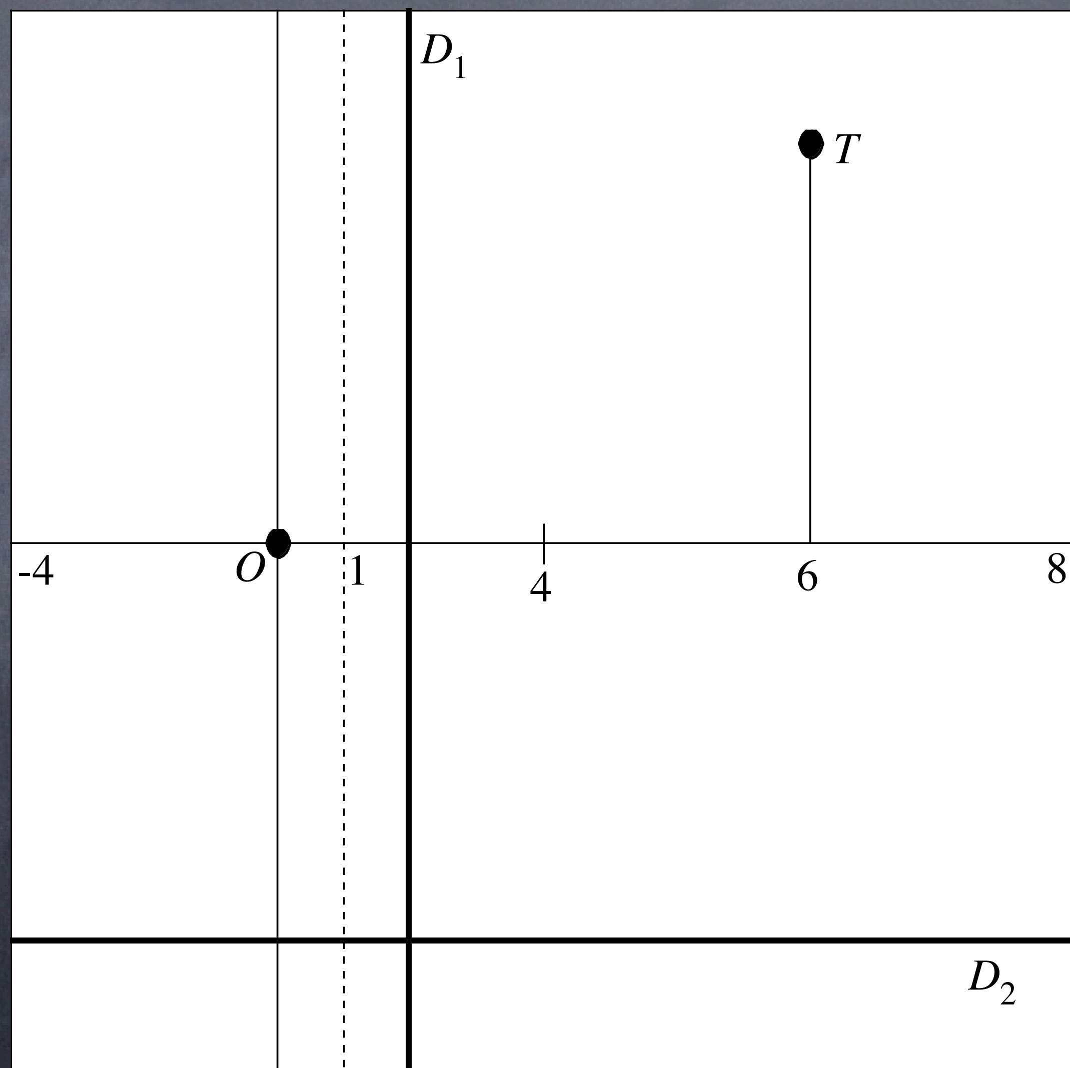

The Algebraic Perspective

The set of constructible numbers under SE&C is the smallest subfield of $\mathbb{C}$ (complex #s) that is closed under square roots.

or...

$\alpha \in \mathbb{C}$ is SE&C constructible if and only if $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2^n$ for some $n \geq 0$. In other words, α is algebraic over $\mathbb{Q}$ and the degree if its minimal polynomial over $\mathbb{Q}$ is a power of 2.

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Origami version:

Let $\alpha \in \mathbb{C}$ be algebraic over $\mathbb{Q}$, and let $L \supset \mathbb{Q}$ be the splitting field of the minimal polynomial of α over $\mathbb{Q}$. Then α is origami constructible from our list of BOOs if and only if $[L : \mathbb{Q}] = 2^a 3^b$ for some integers $a, b \geq 0$.

Oh, but it's worse than that…

Robert Lang's angle quintisection.