6.885	Lecture 23	Dec. 5, 2007

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Straight edge & compass axioms:

1. given two points, draw line connecting them
2. given point as center & line segment as radius, can draw circle
3. can find intersection points among lines & circles

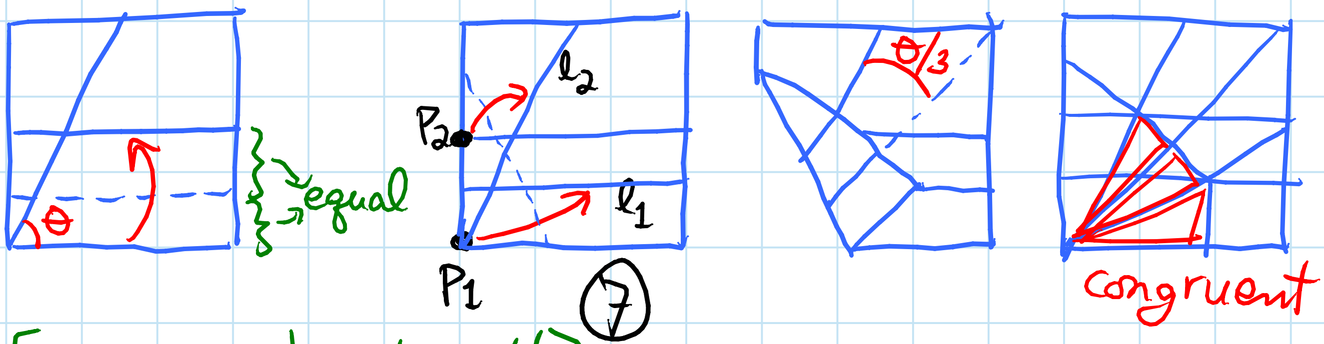
Origami "axioms": Basic Origami Operations (BOO)

1. given points p_1 & p_2, fold line $p_1 p_2$
2. find intersections
3. given point p & line l, fold crease through p & perp. to l
4. given points p_1 & p_2, fold p_1 onto p_2
   $\Rightarrow$ perpendicular bisector
5. given lines l_1 & l_2, fold l_1 onto l_2
   $\Rightarrow$ angular bisector
6. given points p_1 & p_2 & line l,
   fold p_1 onto l with crease thru p_2
   $\Rightarrow$ tangent to parabola focus p_1 directrix l thru p_2
   – doesn't always exist $\Rightarrow$ not really axiom
7. given points p_1 & p_2 & lines l_1 & l_2
   fold $p_1 \to l_1$ & $p_2 \to l_2$ $\Rightarrow$ common tangents ← CUBIC EQUATN

⑦ with P_2 on l_2 ⇒ can do ⑥ (other possibilities) etc.

- can reduce BOOs to ② & ⑦

Angle trisection: (acute angles) [Abe 1980]



[experiment: try it!]

Solving any cubic equation: [Margherita Beloch 1930]

Subproblem:

- let A & B be two points and r & s two lines
- construct square with A & B on opposite sides (extended to lines) & with two adjacent corners on r & s

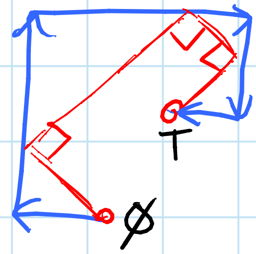
Solution:

- draw parallel d_1 to r with dist(A, r) = dist(r, d_1)
- draw parallel d_2 to s with dist(B, s) = dist(s, d_2)
- fold A → d_1 & B → d_2
- crease gives top of the square
- intersects r & s at corners of square

Now reduces to "method of Lill" for finding real roots of any polynomial equation:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_\emptyset = \emptyset$$

- start at origin $\emptyset$ going $\leftarrow$ $(-x)$
- go a_n, turn $90°$ clockwise
- go a_{n-1}, turn $90°$ clockwise
- etc. $\rightarrow$ turtle position T
- shoot from $\emptyset$ at some angle θ bouncing off the walls at right angles to hit T



$\Rightarrow$ $\tan\theta$ is a real root [Lill 1867]

- proof:
  - shot forms various right triangles with vertex angle θ
  - tan gives ratio of side lengths
  - additive & multiplicative parts

  $\Rightarrow$ get $a_0 + x(a_1 + x(a_2 + x(\cdots + x a_n)))$

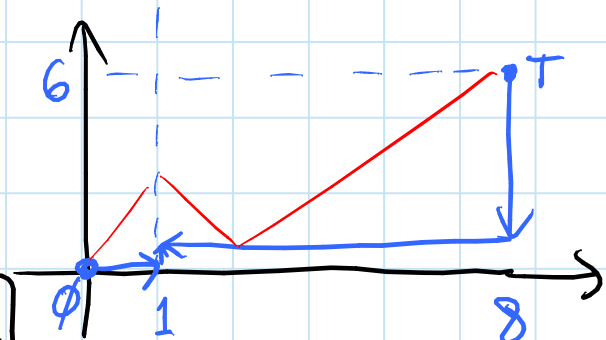
Square construction solves this for degree $n=3$

Cool! [experiment: $z^3 - 7z - 6 = \emptyset$]

$z = -2$

$z = 3$

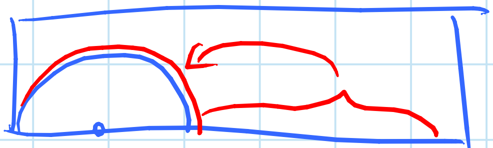
$z = -1$



[can extend to complex case]

Limits:

- straight edge & compass is smallest subfield of $\mathbb{C}$ closed under square roots
  i.e. $\alpha \in \mathbb{C}$ is constructible $\Leftrightarrow [\mathbb{Q}(\alpha):\mathbb{Q}] = 2^n$ for some int. $n \geq 0$
- origami: $\alpha \in \mathbb{C}$ algebraic over $\mathbb{Q}$
  let $L \supset \mathbb{Q}$ be splitting field of minimal polynomial of α over $\mathbb{Q}$
  then α is constructible by BOOs
  $\Leftrightarrow [L:\mathbb{Q}] = 2^a 3^b$ for some ints. $a \& b \geq 0$
- straight creases & one fold at time (simple folds) $\Rightarrow$ BOOs are complete [Lang]
- 2 folds at time $\Rightarrow$ quintisecting angle possible (& don't always unfold) [Lang]
  - working on complete characterization [Alperin & Lang]
  - 2 folds ~ not all quintics (?)
  - 3 folds $\Rightarrow$ all quintics
- curved creases $\Rightarrow$ construct π [Hull]

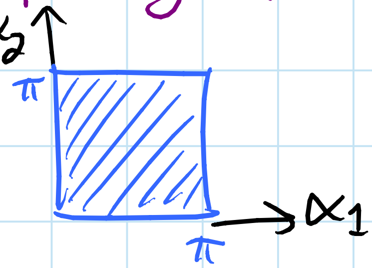


- what about all compass + origami? (OPEN) [Aviv Ovadya]

Configuration space of flat-foldable single-vertex CPs (θ_i's)

[new paper by Hull]

- degree 4 $\Rightarrow$ square (α_1, α_2 determine α_3, α_4)



- degree 6 $\Rightarrow$ 4-polytope
- in general:

$$f_K = \sum_{i=0}^{k} \binom{n}{i+1}\binom{n}{k-i+1} = \dots$$

& polytope = Minkowski sum of two (n-1)-dimensional simplices