6.885	Lecture 21	Nov. 27, 2007

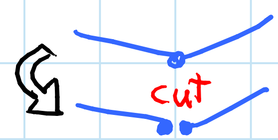
Interlocked 3D chains: [Demaine, Langerman, O'Rourke, Snoeyink 2002 & 2003]

- smallest locked chain has 5 bars (knitting needles), 6 if closed [Cantarella & Johnston 1996]
- can we get away with fewer bars (per chain) if we have more than one chain?

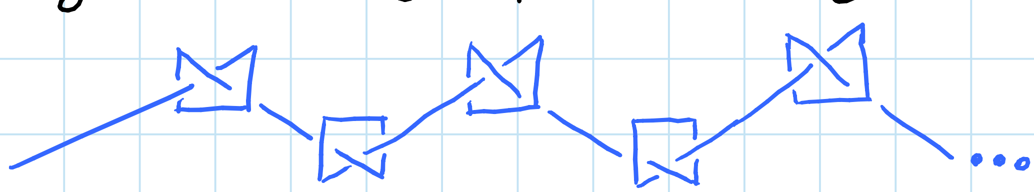
Interlocked set of chains = cannot be separated arbitrarily far by motion avoiding intersection

Motivation: Lubiw's problem [2000]

$\boxed{\text{OPEN}}$: what is the minimum number of vertex "cuts" that suffice to "unlock" any n-bar open chain?



- certainly $\geq \lfloor \frac{n-1}{4} \rfloor$ by repeated knitting needles:



- $\lfloor \frac{n-1}{4} \rfloor$ cuts guarantee that each piece is unlocked but two or more pieces might be interlocked
- best upper bound: $\lfloor \frac{n-3}{2} \rfloor$

Main results on interlockability:

OPEN CHAIN	OPEN CHAINS			CLOSED CHAINS		
	2	3	4	3	4	5
2	sep.*	sep.*	sep.*	sep.	sep.	sep.
3	sep.*	sep.*	interlock	sep.	interlock	interlock
4	sep.*	interlock	interlock	interlock	interlock	interlock

sep.*: even with any number of 2-bar open chains

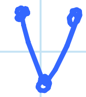
Also: three 3-bar open chains interlock
⇒ complete characterization of interlocking of sets of open chains EXCEPT:

OPEN: smallest k for which 2-bar open chain interlocks with a k-bar open/closed chain?

- open 16-chain
  [Glass, Langerman, O'Rourke, Snoeyink, Zhong 2004]
- open 11-chain, conjectured optimal
  [Glass, Lu, O'Rourke, Zhong 2006]

No finite set of 2-chains can interlock

"hairpins"



... even if 2-chains are held rigid!

- explosion motion: [Dawson 1984; de Bruijn 1954]
  - scale 3D space from origin by factor $1+t$, time $t \geq 0$ (point $p \mapsto (1+t)p$)
  - preserves non-intersection (affine transform) but not the edge lengths: all too long
  - shorten bar lengths to original lengths as time t proceeds
  - subset of scaled version $\Rightarrow$ no intersection
  - chains separate as $t \to \infty$
- key property: 2-chains are <u>starshaped</u> sets (union of line segments from one point)

Two open 3-chains can't interlock

... even with finitely many 2-chains added in

- perturb chains so that no nonincident bars are coplanar & no three vertices are collinear
- find plane parallel to two middle bars such that middle bars are on opposite sides
- re-orient to make it the xy plane
- perturb chains so that all z coords. distinct
- explode just in z: $(x, y, z) \mapsto (x, y, (1+t)z)$, $t \geq \emptyset$
- middle bar lengths preserved
- other lengths increase ~ trim appropriately
- preserves no intersection
- all chains separate vertically

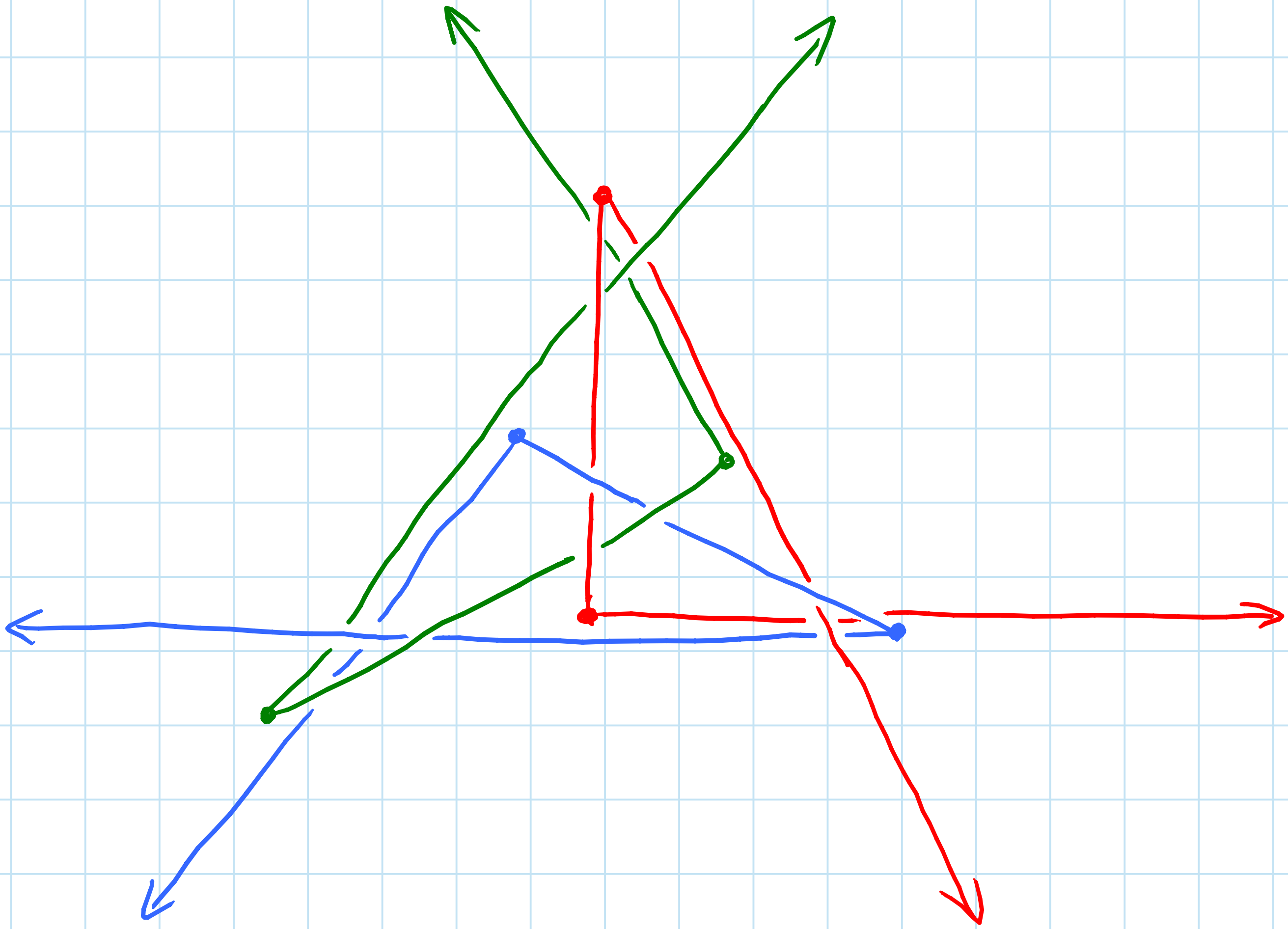
4-chain + 2-chains can't interlock: similar

- re-orient to put middle bars in xy plane
- explode just in z

$\Rightarrow \lfloor \frac{n-3}{2} \rfloor$ upper bound on Lubiw's problem

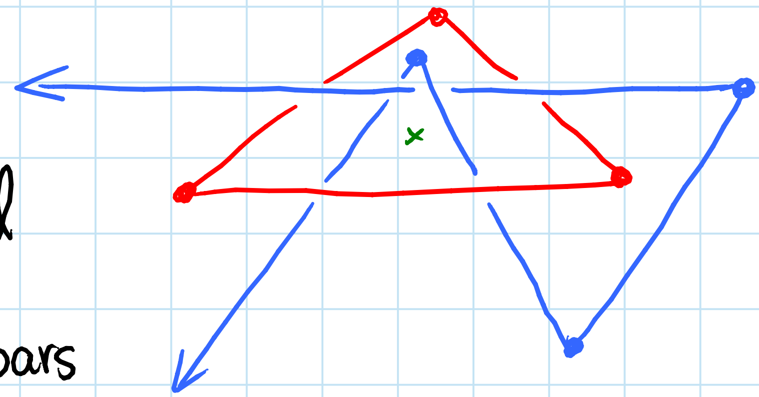
- cut 4th joint, then every other joint

Three 3-chains can interlock:

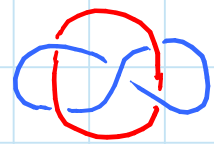


4-chain can interlock with triangle: (closed 3-chain)

- view Δ as fixed
- suppose Δ has circumcircle centered at origin, radius r
- let $R = r + \sum$ middle bars of 4-chain



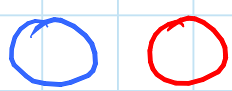
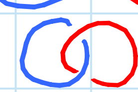
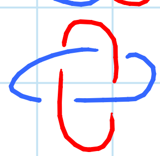
- require end bars of 4-chain to have length $\geq 20R$
- as long as endpoints stay outside ball centered at origin & radius $15R$, and middle bars stay inside ball, can connect endpts. with rope outside ball:



5^2_1

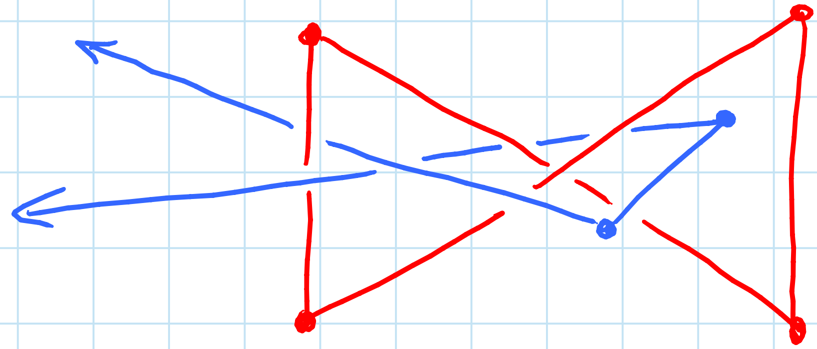
- consider first violation: two cases
  1. if middle vertex about to exit, all middle vertices are $\geq 14R$ away from origin
  2. if endpoint about to enter, adjacent vertex is $\geq 5R$ away from origin $\Rightarrow$ others are $\geq 4R$ away

$\Rightarrow$ only end bars can strike the Δ's interior

- three cases:
  1. no end bars strike Δ:  0^2_1
  2. one end bar strikes Δ:  2^2_1
  3. both end bars strike Δ:  4^2_1 or 0^2_1
     (with some argument)
- none of these match original topology (5^2_1) & topology can't change before such an event. $\square$

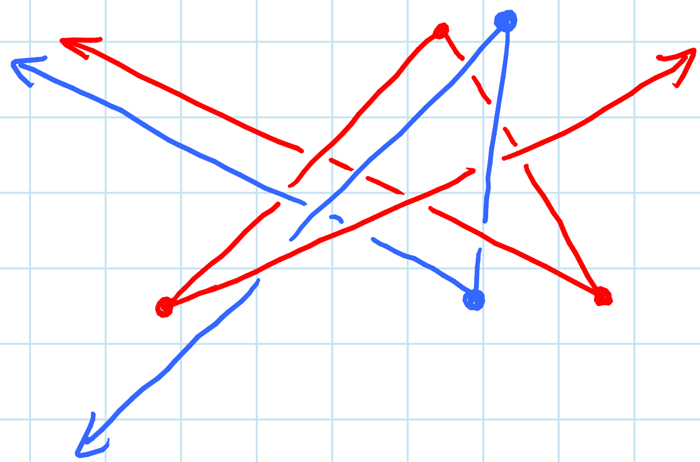
3-chain can interlock with quad: (closed 4-chain)

- another topological argument



3-chain can interlock with 4-chain:

- topological argument difficult (impossible?) for two open chains
- instead use **geometric argument** about bars striking faces of convex hull of middle joints



Variations:

- rigid chains (like 2-chain result)
- fixed-angle ("revolute") chains

Summary of results:

		2		3			4			5
		flex.	rigid	flex.	fix.ang.	rigid	flex.	fix.ang.	rigid	rigid
2	flex.	−	−	−	−	−	−	−	−	+
	rigid	−	−	−	−	+	+	+	+	+
3	flex.	−	−	−	−	+	+	+	+	+
	fix.ang.	−	−	−	+	+	+	+	+	+
	rigid	−	+	+	+	+	+	+	+	+
4	flex.	−	+	+	+	+	+	+	+	+
	fix.ang.	−	+	+	+	+	+	+	+	+
	rigid	−	+	+	+	+	+	+	+	+
5	rigid	+	+	+	+	+	+	+	+	+