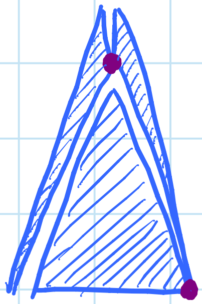
6.885	Lecture 20	Nov. 26, 2007

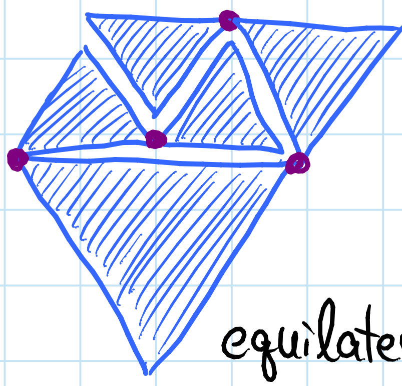
Locked & unlocked chains of planar shapes:

rigid objects in place of bars [Connelly, Demaine, Demaine, Fekete, Langerman, Mitchell, Ribó, Rote 2006]

- simple locked examples:
  [M. Demaine 1998]



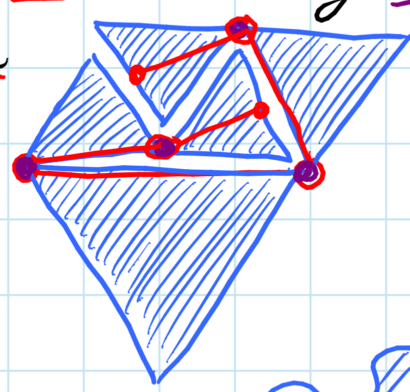
3 triangles



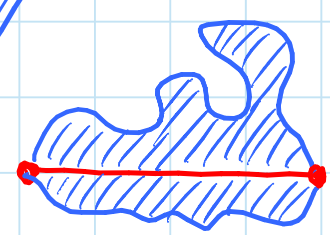
equilateral Δs

Adorned chain: view shapes as adornments attached to bar connecting hinges

- underlying chain linkage
- some flexibility in first & last shape

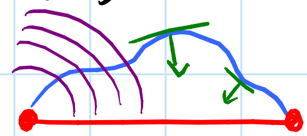


Adornment = shape + base

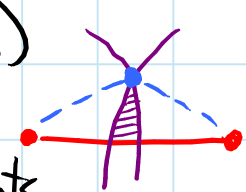


- shape = simply connected compact planar region
- base = line segment connecting 2 boundary pts.
- require base to be contained in shape

Slender adornment = walking along boundary from one base endpoint to the other monotonically increases distance to former (& decreases distance to latter)



= all inward normals hit the base (for piecewise-differentiable shapes)

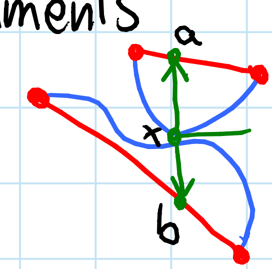


= (possibly infinite) union of half-lenses: intersection of disks centered at base endpts. & halfplane on one side of base

(⇒ can define slender hull = union of half-lenses thru every point in adornment)

Slender ⇒ not locked: expansive motion of underlying chain linkage preserves nonintersection of slender adornments

Intuition: consider two touching adornments

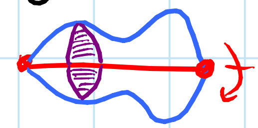


- draw collinear inward normals from touching point x
- resulting points a & b expand (vertices expand ⇒ points on bars expand)

⇒ two copies of x locally expand

- in reality, this argument is tricky: can stay equal, to first order
- possible with strict expansiveness [see SoCG 2006 proof]

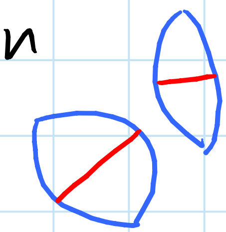
Symmetric case: adornments reflectionally symmetric about their bases



⇒ slender adornment = union of lenses

Stronger result for this case:
instantaneous expansive motion of underlying chain linkage preserves nonintersection of slender adornments

Proof: take any two lenses of different adornments
– nonintersecting before the motion
i.e. four disks have empty intersection

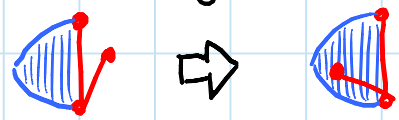


Kirszbraun's Theorem: [1934]
if we instantaneously translate n disks with an empty n-way intersection according to an expansive motion on their centers, then they still have empty intersection

(annoying detail: Kirszbraun's disks include their boundary, but our disks might kiss — but Kirszbraun's Theorem holds for disks excluding their boundaries, by taking limits of slightly smaller disks) □

If initially intersecting, can show area of union only increases [Bezdek & Connelly 2002 ~ Kneser-Poulsen conj.]

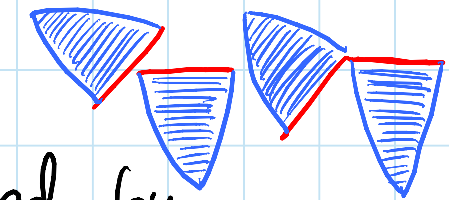
Proof that slender $\Rightarrow$ not locked: (general case)

(– not true for instantaneous: )

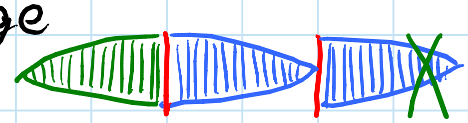
– take any 2 half-lenses of different adornments

– consider transition time from not intersecting to intersecting $\Rightarrow$ touching

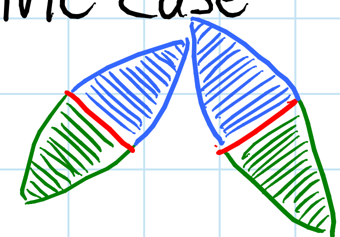
– 3 types of touching:



① bases of both
  – nonintersection guaranteed by underlying chain linkage



② base of one
  – can add symmetric lens of other, & just consider base of first (X)
  – no intersection by symmetric case

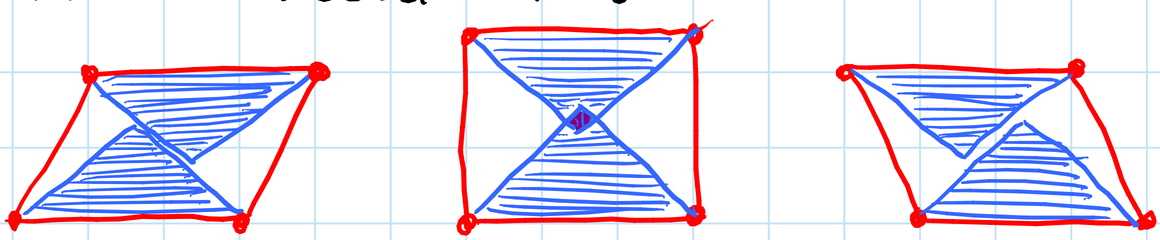


③ base of neither
  – can add symmetric lens of both
  – again no intersection by symmetric case □

Carpenter's rule theorem $\Rightarrow$ straighten/convexify any slender-adorned (non-self-touching) chain

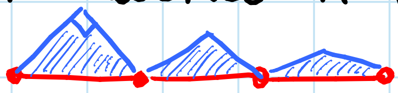
$\Rightarrow$ connected config. space of open chains

– not true of closed chains:

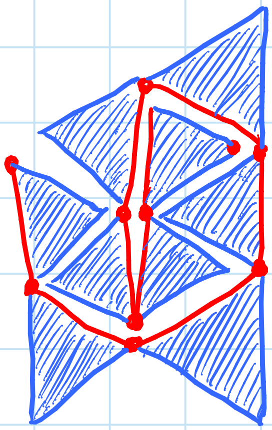


OPEN: which adornments never lock in a chain?
(like slender)

Triangles: not locked if angles opposite base $\geq 90°$
(right or obtuse)



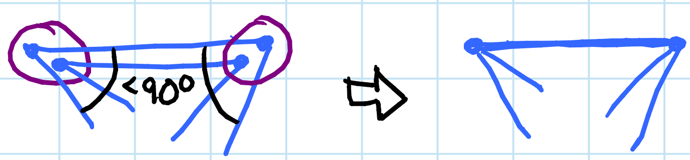
— locked $\approx$ identical equilateral triangles:



— can stretch/shrink in y coord. to make locked example with any angle $< 90°$

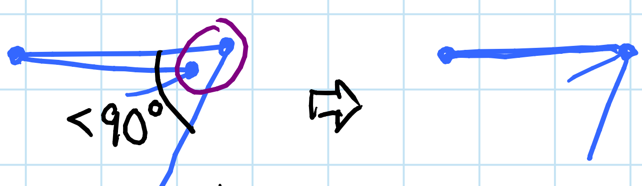
Proof: show self-touching version rigid
$\Rightarrow$ strongly locked [Lecture 5]

— Rule 1:



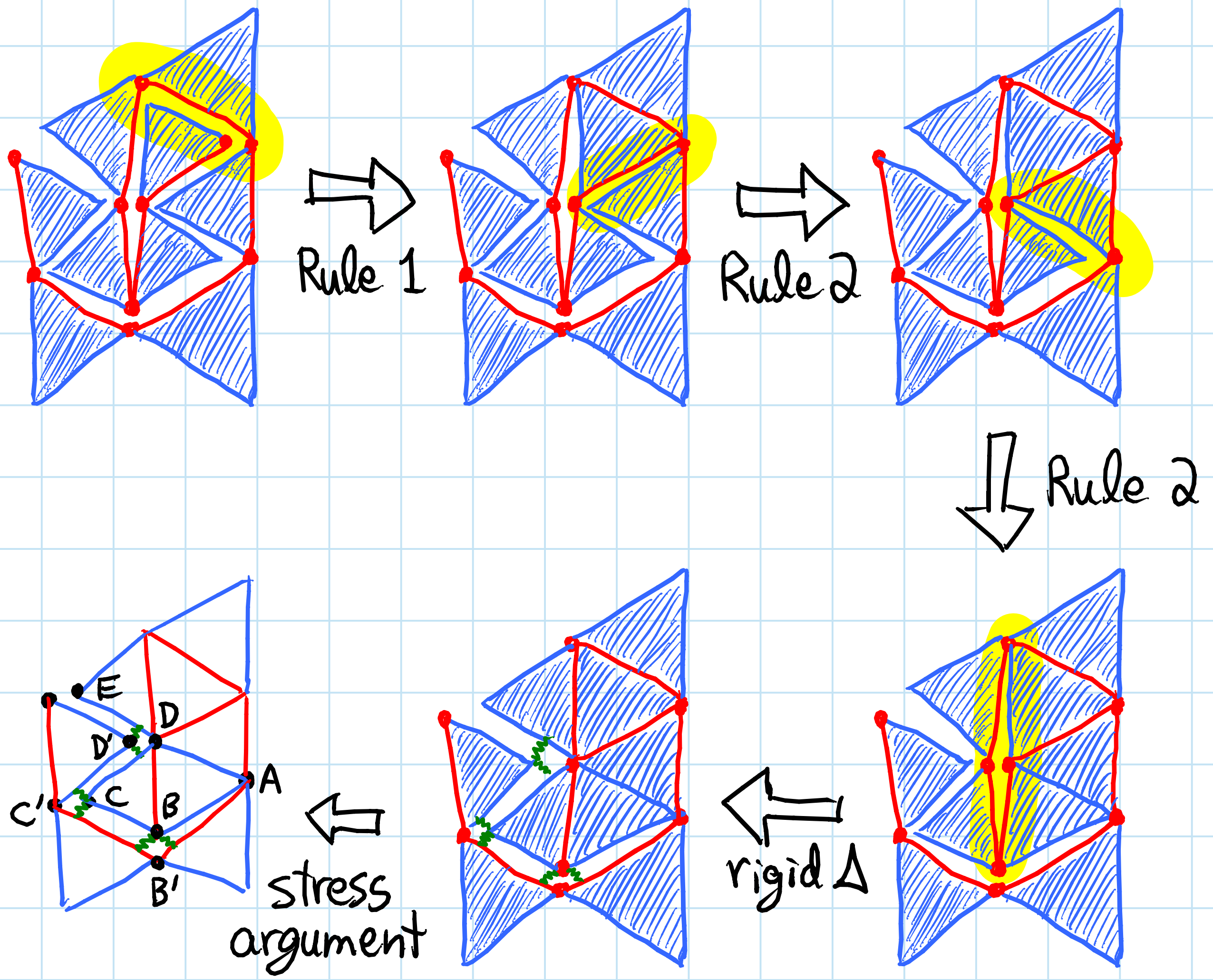
because inner bar must stay pinned against outer bar until an angle $\geq 90° \Rightarrow$ positive time

— Rule 2:



by same argument

Locked triangles proof: (cont'd)



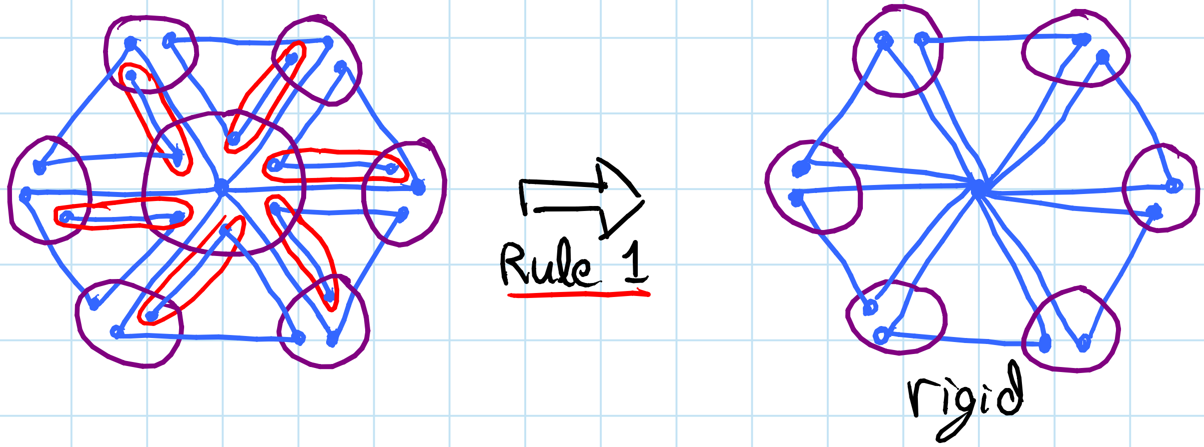
- clearly rigid if zero-length struts were bars
- set $s(AB) = -s(AB') < 0 \Rightarrow A$ in equilibrium
- set $s(BC) = s(AB) = -s(B'C') = -s(AB') < 0$

$\Rightarrow$ force on B, B' vertical $\Rightarrow$ in equilibrium if set $s(B, AB') = s(B, B'C') < 0$ appropriately

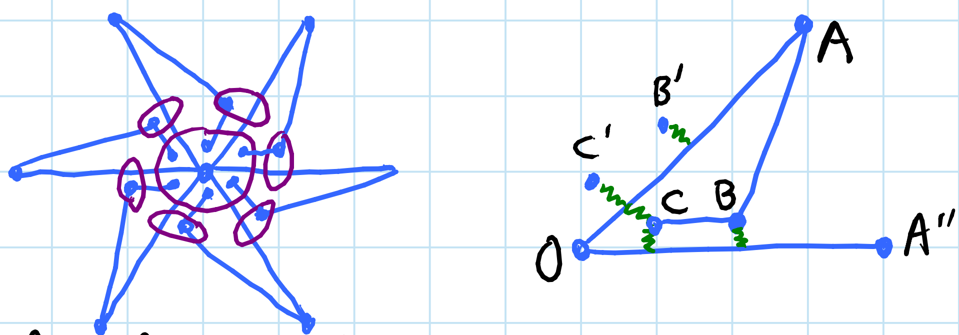
- set $s(C'D'), s(D', DC), s(D', DE) < 0$ unique up to scale to put D' in equilibrium; scale very small
- set $s(CD) = -s(C'D') \Rightarrow D$ in equilibrium (inverse of D')
- $s(BC) < 0$ dominates $s(CD) \Rightarrow$ can set $s(C, C'B')$ & $s(C, C'D') < 0$ to put C (& hence C') in equilibrium $\square$

Locked tree arguments: [some new realizations]

- Rule 1 immediately rigidifies "triangle tree":



Stress argument for Biedl et al. tree: [Connelly, Demaine, Rote 2002]



- clearly rigid if zero-length struts were bars
- $s(CB), s(C, AO), s(C, A''O) < 0$ unique up to scale to put C in equilibrium

$\Rightarrow s(BA), s(B, A''O) < 0$ uniquely determined for B

- symmetric on petals $\Rightarrow$ O in equilibrium
- force so far on A must be parallel to A (because forces must induce zero rotation around O)

$\Rightarrow$ can set $s(AO) > 0$ to put A in equilibrium □