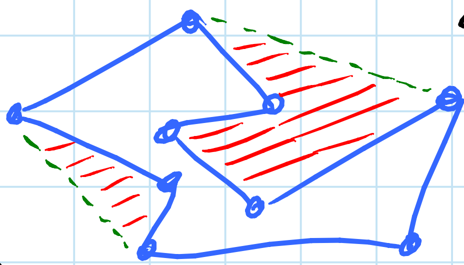
6.885	Lecture 19	Nov. 18, 2007

Flips & friends: (recall from Lecture 5)

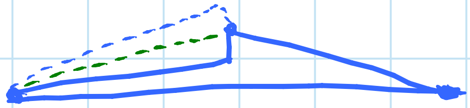
Pocket of 2D polygon = region outside polygon & inside convex hull



Pocket lid = convex-hull edge

Flip = reflect pocket through its lid
= rotate 180° through 3D around the lid
- avoids self-intersection (line of support)
- increases area

"Erdős-Nagy" Theorem: [posed by Erdős 1935]
any polygon always convexifies after finite flips, no matter how flip sequence is chosen
- but can be arbitrarily many: [Joss & Shannon 1973]

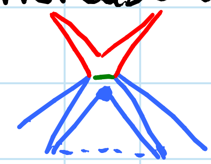


- $\boxed{\text{OPEN}}$: bound # flips in n & r = max. dist./min. dist
  - pseudopolynomial? [Overmars 1998]

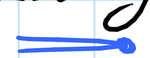
"Proofs" of Erdős-Nagy Theorem: [Demaine, Gassend, O'Rourke, Toussaint 2007]

knowledge

- Nagy 1939 — flawed: "$P^0 \subseteq C^0 \subseteq P^1 \subseteq C^1 \subseteq \cdots$"
  (used to "prove" limit polygon convex)
- Reshetnyak 1957 — correct (though somewhat imprecise)
- Yusupov 1957 — flawed: "limit convex else flip" & more subtle error
- Bing & Kazarinoff 1959 — correct (though somewhat terse)
- Wegner 1993 — flawed: "move vertex ⇒ increase area by incident Δ"



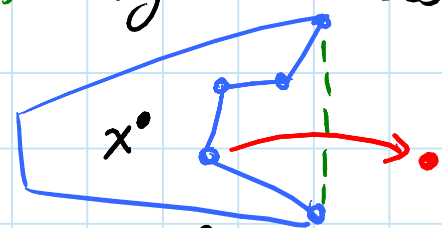
- all ↑ Grünbaum 1995 — omission: why limit polygon is convex
- all ↑ Toussaint 1999/2005 — flawed: "limit convex else flip"
- all ↑ Demaine et al. 2007 — generalization to self-crossing assuming no "hairpins":



Proof of "Erdős-Nagy" Theorem: [Bing & Kazarinoff 1959] & [CCCG 2006]

consider an infinite flip sequence

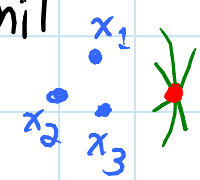
① distance from a vertex to fixed point x inside the polygon (remains so) only increases
- pocket lid is Voronoi diagram of old & new



② each vertex approaches a unique limit
- apply ① to three noncollinear points x_1, x_2, x_3 inside the polygon
- distances from vertex ≤ perimeter of polygon/2

⇒ distances converge

⇒ vertex approaches intersection of 3 circles



③ turn angle at each vertex converges
- by ②, 3 vertices defining the angle converge
- by ①, vertices do not get closer to each other
- rest by continuity

④ vertex moves infinitely ⇒ asymptotically flat
- each move negates sign of turn angle ⇒ → ∅

⑤ contradiction
- eventually asymptotically pointed vxs. stop moving

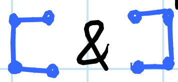
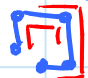
⇒ attain limit convex hull, but about to flip! □

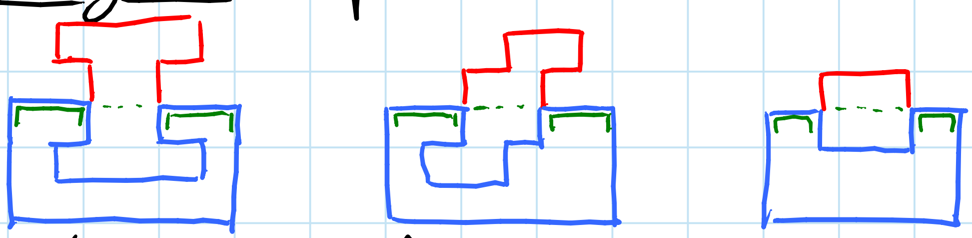
Flipturn: rotate pocket 180° in 2D around lid midpoint

- at most $n!$ configurations [Joss & Shannon 1973]
- always $O(n^2)$ flipturns [Aichholzer et al. 2002; Ahn et al. 2000 (diff. model)]
- sometimes $\Omega(n^2)$ flipturns [Biedl 2004]
- final polygon & location determined
- NP-hard to find longest flipturn sequence
- OPEN: finding shortest flipturn sequence?

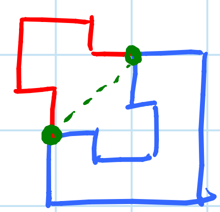
[Aichholzer et al. 2002]

Orthogonal polygons: $< n$ flipturns

- count brackets: 
  - allow overlap  $\Rightarrow \le n$ brackets
- claim # brackets never decreases
  (13-case analysis)
- orthogonal flipturn kills two brackets:



$\Rightarrow \le n/2$ orthogonal flipturns

- diagonal flipturn kills two vertices: 

$\Rightarrow < n/2$ diagonal flipturns

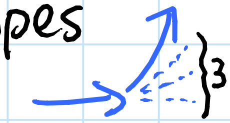
$\Rightarrow < n$ total $\square$

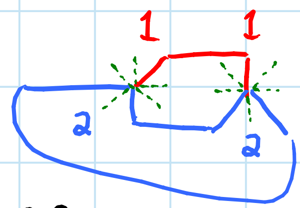
OPEN: $n - O(1)$ flipturns ever possible?

- best example requires $\frac{5}{6}n - O(1)$

Flipturns: (cont'd)

General polygons: $\leq ns$ if s distinct slopes

- discrete turn angle $= 1 + \#$slopes between 

- measure total discrete turn angle: 

  - nondegenerate flipturn decreases by ≥ 2
  - degenerate flipturn doesn't change
- also count brackets: 
  - nondegenerate flipturn increases by ≤ 2
  - degenerate flipturn decreases by ≥ 2
- potential function $=$ total disc. angle $+ \frac{1}{2}\#$brackets
  - any flipturn decreases by ≥ 1
  - initially $\leq n(s-1) + n = ns$ $\square$

Algorithms for Alexandrov's Theorem: (again)

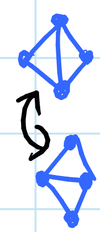
- constructive proof of [Bobenko & Izmestiev 2006] is not technically an algorithm
- need to approximate differential equation by taking "small enough" steps
  - can change K's only slightly & maintain r's
  - triangulation must flip to maintain convexity
- when is step size small enough?
- must also be large enough to guarantee termination

OPEN: finite algorithm based on this construction

OPEN: [Carola Wenk – Nov. 2007]
can you compute times at which triang. would flip & set step size = min?

OPEN: (pseudo)polynomial-time algorithm?

OPEN: [Joseph O'Rourke – Nov. 2007]
how many flips can there be?



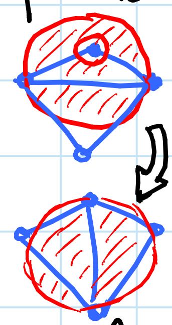
OPEN: [Boris Aronov – Nov. 2007]
can a quad flip & later flip back?

- also the first step: Delaunay triangulation
  - definitely finite
  - is it (pseudo) polynomial?

Delaunay triangulation algorithm: [Bobenko & Springborn 2006]

- start from some geodesic triangulation
  e.g. repeatedly add noncrossing shortest paths
- while some edge not locally Delaunay:
  - flip it

some circumcircle of edge contains no other vxs



Lemma: if not locally Delaunay then flippable:
- two distinct triangles (topology)
- intrinsically convex quad. (geometry)

Lemma: Delaunay flip decreases the sum of areas of Δ circumcircles [Telly – PhD 1992]
(& Musin's harmonic index) (computation)

Lemma: finitely many geodesics from p to q of length $\leq L$, for any $L \geq 0$
(nontrivial)

$\Rightarrow$ finitely many triangulations with Σ areas $\leq A$
$\Rightarrow$ finitely many flips

$\boxed{\text{OPEN}}$: (pseudo)polynomial if initially shortest paths?