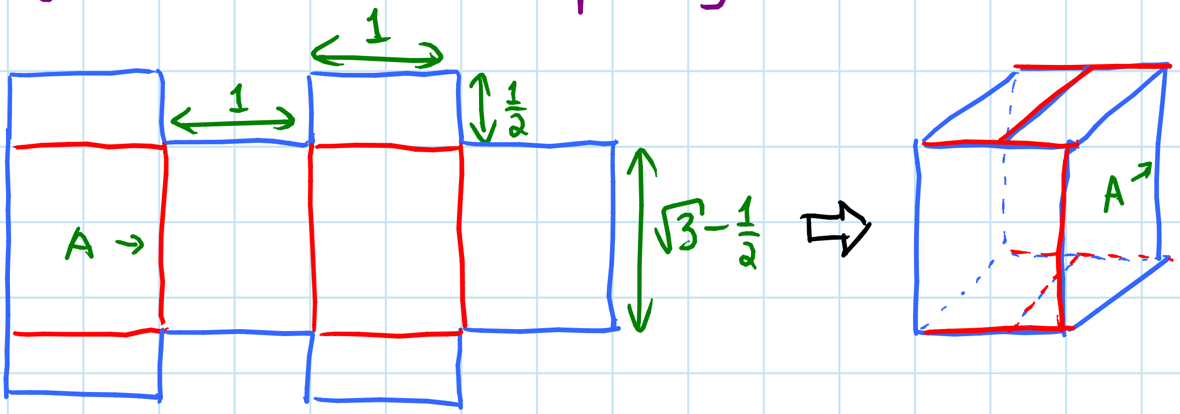
6.885 | Lecture 14 | Oct. 29, 2007

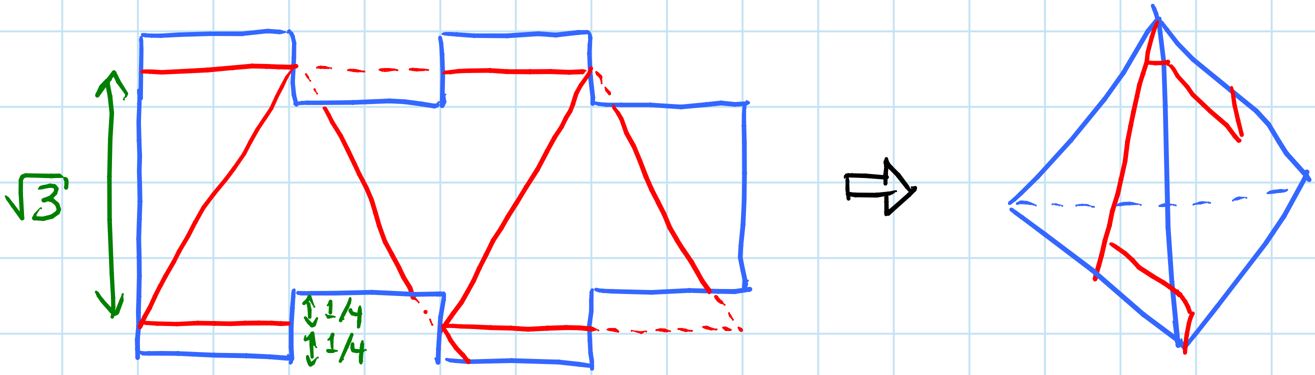
Polyhedron unfold/refold "dissection":

OPEN: which polyhedra P, Q are connected by
P → (general) unfolding → (general) gluing → Q?
or a sequence of unfolding, gluing, ...?
— true for any two Platonics? [M. Demaine, 1998]

Interesting examples:

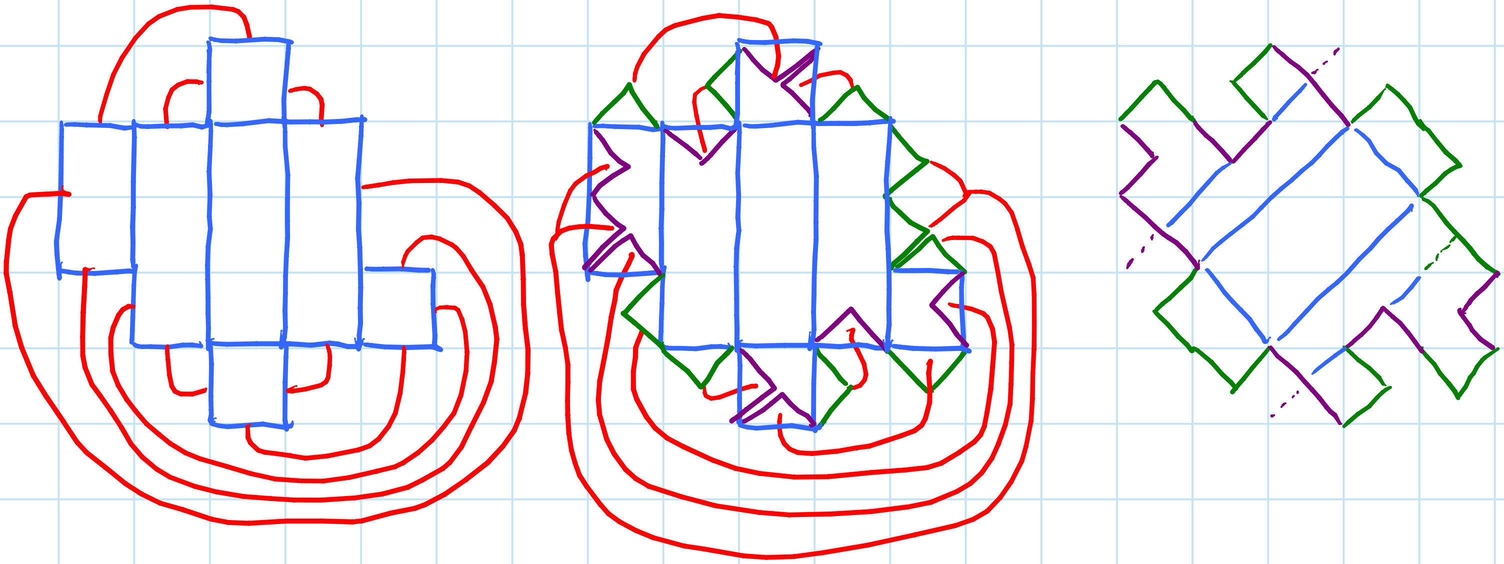
— regular tetrahedron → $1 \times 1 \times \underbrace{1.232}_{\sqrt{3}-\frac{1}{2}}$ box possible
[Hirata 2000; GFALOP p.425]





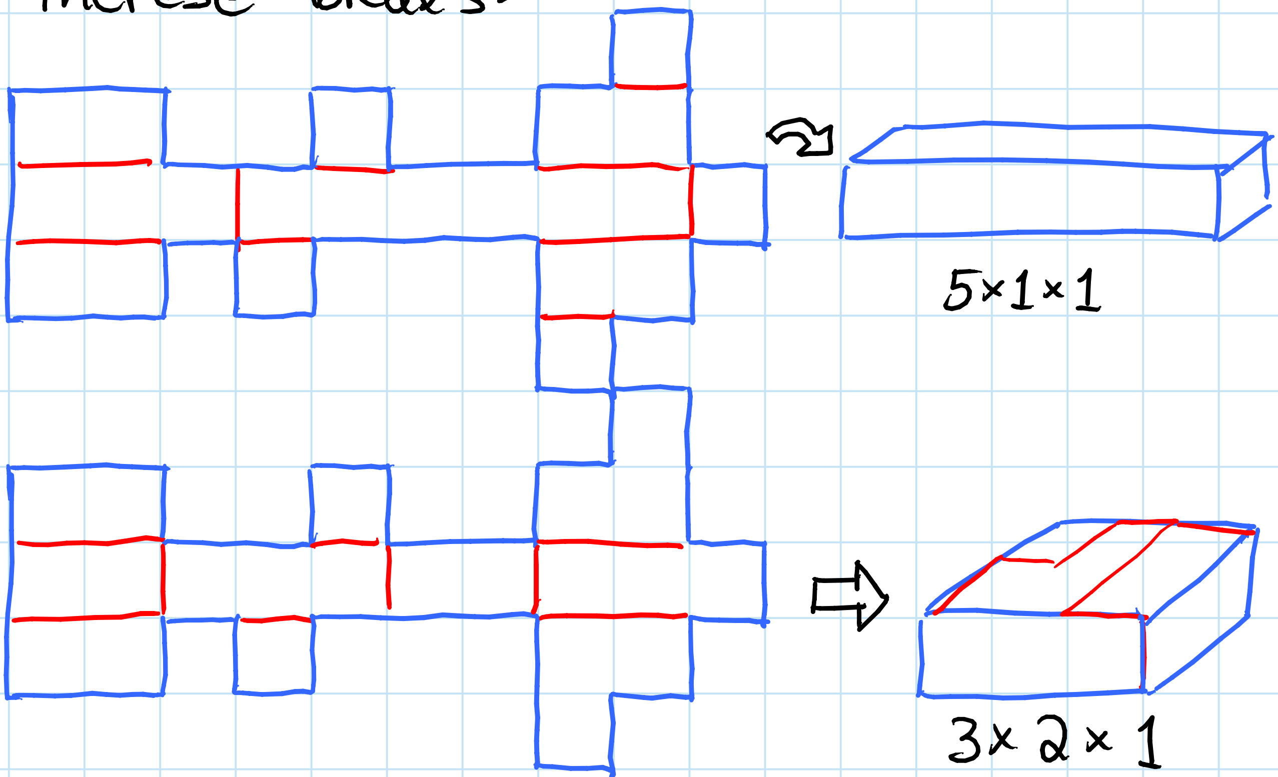
Box → box: [Biedl, Chan, Demaine, Demaine, Lubiw, Munro, Shallit 1999]

- Timothy Chan's solution: [GFALOP p. 425]



$\sqrt{2}\times\sqrt{2}\times3\sqrt{2}$ → add tabs → 1×2×4 box

- Therese Biedl's:



- also 8×1×1 → 5×2×1
- **OPEN**: when is a×b×c → d×e×f possible?

Orthogonal polygons vs. polyhedra:

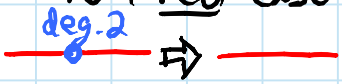
Can an orthogonal polygon with orthogonal creases fold into nonorthogonal polyhedra?

- answer depends on allowed genus

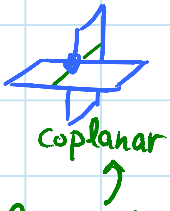
Genus	NonOrthog.?	Reference
0	NO	[Donoso & O'Rourke 2002]
1 & 2	NO	[Biedl et al. 2002]
3 – 5	OPEN	
6	YES	[Biedl et al. 2002]
≥7	YES	[Donoso & O'Rourke 2002]

Proof sketch that genus ≤ 2 ⇒ impossible:

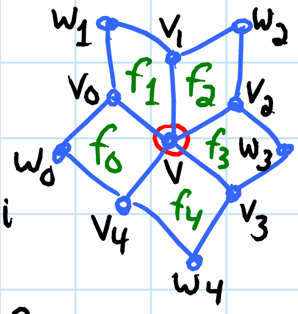
- edge green if dihedral multiple of 90°; red else
- focus on red subgraph; coallesce



- faces not planar, but face angles still mult. of 90° & dihedral angles still not mult. of 90°
- vertices have degree ≥ 4; deg. 4 special:



- claim: $V \le 8(g-1) - \max\{D_{\neq 5}, F_{\geq 5}/2\}$
  (proof uses Euler's Theorem + analysis of special vxs.)
- degree ≥ 4 & $V \geq 1 \Rightarrow V \geq 5$
- claim $\Rightarrow D_{\neq 5} \leq 3 \Rightarrow D_5 \geq 2 \rightarrow v$
- $V \geq 6 \Rightarrow F_{\geq 5} \leq 4 \Rightarrow$ some f_i degree 4 $\rightarrow w_i$
- $V \geq 7 \Rightarrow F_{\geq 5} \leq 2 \Rightarrow \cdots \Rightarrow$ all f_i degree 4
- $w_i \neq w_{i+1} \Rightarrow \geq 3$ distinct $\Rightarrow V \geq 9$ but $V \leq 8$ □



Smooth Alexandrov: [Pogorelov 1973]
every convex metric, topologically a sphere,
is realized by a unique convex surface,
possibly degenerating to flat doubly covered convex shape
- proof idea: take limits of polyhedral approximation

D-forms: [Tony Wills; Pottmann & Wallner 2001]

- take two convex smooth shapes of the same length
- identify two boundary points of two shapes
- glue around from there
- smooth Alexandrov ⇒ get convex surface

OPEN: is a D-form always the convex hull of a space curve?

OPEN: are interiors of smooth shapes always free of creases?

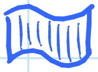
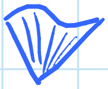
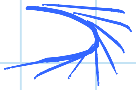
OPEN: which space curves are "ribbon curves"
i.e. the edge of uncreased paper rectangle?
[O'Rourke 2004]

- not all C^∞ curves [Robert Dawson 2005]

Curved-crease origami:

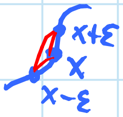
- Bauhaus [1920s–30s]
- Demaine & Demaine [MOMA 2008]
- Ron Resch [unpublished]
- David Huffman (1925–1999) (of Huffman codes)
- Richard Sweeny
- Jeannine Mosely
- Philip Chapman-Bell
- a few other artists (surprising few)

Developable surface: zero curvature & uncreased

⇒ generalized cylinder , cone , or tangents of space curves 

- in all cases, ruled: decompose → generating lines

Local facts: consider space curve formed by crease in flat paper [Resch & Huffman 1976]

- osculating plane at any point x
  ↳ plane best fitting curve at & near x 

  bisects two tangent planes of surface on either side of crease
- if curve is planar then generating lines on either side form equal angle with plane
- every space curve can be achieved by a crease in flat paper

Wrapping smooth surfaces with flat paper:

[Demaine, Demaine, Iacono, Langerman 2007]

- impossible with finite number of creases to make all points have nonzero curvature
- *idea*: allow folding to shrink some intrinsic distances on paper (*contractive mapping*)
  - simulate by crinkling paper:

Burago & Zalgaller Theorem: [1996]

any contractive C^2-immersion of a polygon (or more generally, a polyhedral metric) admits a C^0-approximation (within $\varepsilon\ \forall \varepsilon > 0$) by isometric piecewise-linear C^0-immersions

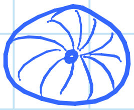
- also, noncrossingness is preserved

Stretched path = isometrically folded (unshrunk) path

- optimal wrapping should have one; else scale

Stretched wrapping = stretched path between all two points

Source wrapping of convex surfaces:



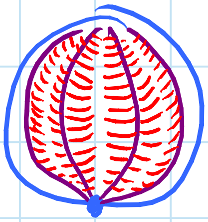
- stretched paths along all shortest paths from x
- cut at ridge tree / cut locus
- e.g. unit sphere ⇒ disk of radius π, area π^3, perimeter $2\pi^2$

Strip wrapping:

- e.g. unit sphere ⇒ area $\to 4\pi$, perimeter $\to \infty$

Wrapping smooth surfaces with flat paper: (cont'd)

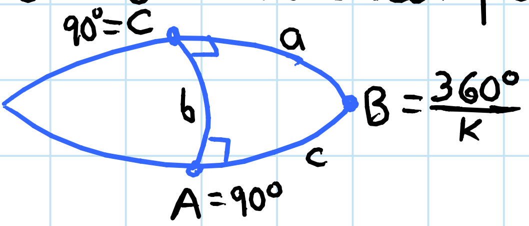
Petal wrapping of sphere:

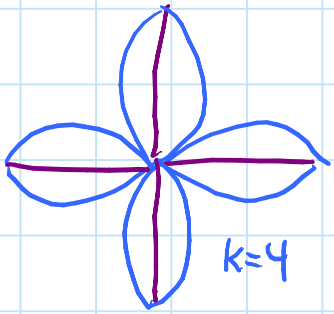


- k stretched paths from south to north pole
- perpendicular stretched paths till meet another

$\Rightarrow$ $360°/k$ "orange peel" for each primary path

- depth-2 tree of stretched paths





- Spherical Law of Cosines $\Rightarrow$

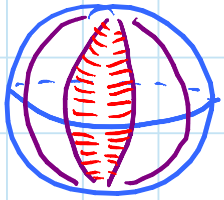
$$\cos C = -\underbrace{\cos A}_{0} \cos B + \underbrace{\sin A}_{1} \sin B \cos c$$

$$\Rightarrow \sin C = \sqrt{1 - \sin^2 B \cos^2 c}$$

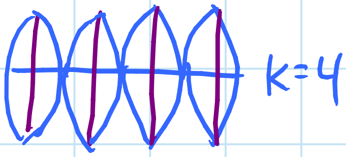
- Spherical Law of Sines $\Rightarrow$ $\dfrac{\sin B}{\sin b} = \dfrac{\sin C}{\sin c}$

$$\Rightarrow b = b(c) = \arcsin \frac{\sin c}{\sqrt{\frac{1}{\sin^2 B} - \cos^2 c}}$$

Comb wrapping: depth-3 tree



- stretched path around equator
- k paths from equator to north pole, ditto for south pole
- perpendicular stretched paths
- same petals, different gluing



Wrapping smooth surfaces with flat paper: (cont'd)

Both petal & comb wrappings: area → 4π
perimeter → ∞

Real Mozartkugel wrappings:

- square containing 4-petal [Fürst]
  - $\sqrt{2}\pi \times \sqrt{2}\pi$
  - $\Rightarrow$ area $2\pi^2$, perimeter $8\pi/\sqrt{2} \approx 5.7\pi$
- rectangle containing 4-comb [Mirabell]
  - $\pi \times 2\pi$
  - $\Rightarrow$ area $2\pi^2$ (!), perimeter $6\pi >$

Better Mozartkugel wrappings:

- equilaterial triangle containing 3-petal has area $1.9983\pi^2$ ($\approx 0.1\%$ improvement)

- packing 3-petals $\Rightarrow$ $1.6033\pi^2$ area each
- packing k-combs $\Rightarrow$ $1.3333\pi^2$ area each
- vs. optimal: $4\pi = 1.2732\pi^2$

OPEN: what is the best area for given perimeter? (Pareto curve)

OPEN: what is the minimum possible perimeter?

OPEN: what about smooth surfaces other than the sphere?