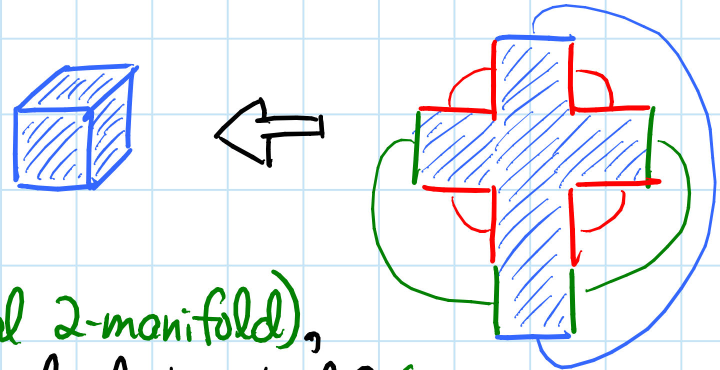
6.885	Lecture 12	Oct. 22, 2007



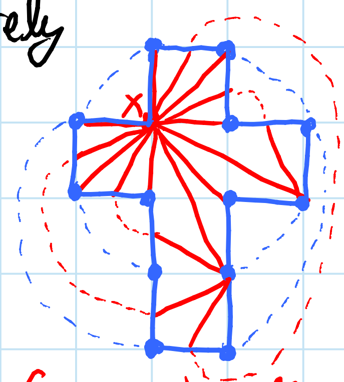
Folding polyhedra:

Decision problem:
given a polygon
(or connected metric polygonal 2-manifold),
can its boundary be glued to itself (in pairs of intervals)
such that resulting surface can be folded
into exactly a convex polyhedron?
↳ no multiple layers like origami

Enumeration problem: list all gluings & foldings
Combinatorial problem: how many can there be?

Why convex polyhedra?
$\boxed{\text{OPEN}}$: if the goal is any nonconvex polyhedron
without boundary, is the answer YES
for all polygons? [O'Rourke 2004]

Alexandrov gluing: polygon + gluing induce a metric
by shortest-path lengths between all pairs of points
- metric is polyhedral: all but finitely many points have zero curvature
- metric is convex if all points have zero or positive curvature
- metric is topological sphere if gluing noncrossing



shortest paths from x to all vxs.

Alexandrov's Theorem: [1941; English book 2005]

every convex polyhedral metric, topologically a sphere, is realized by a unique convex polyhedron (possibly degenerating to doubly covered flat polygon)

Proof sketch:

Uniqueness: draw all shortest paths between pairs of vxs.

- includes all edges of any polyhedral realization

$\Rightarrow$ faces between mesh of paths are rigid

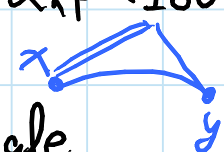
- Cauchy's Rigidity Theorem $\Rightarrow$ unique convex realiz.

Existence: induct on n = #vertices

- base case: $n \leq 4$ (double triangle or tetrahedron)
- total curvature of all vertices $= 720° = 4\pi$

  [Descartes' Theorem; conseq. of Gauss-Bonnet Formula]
- $n \geq 5 \Rightarrow 2$ vertices x, y have curvatures $\alpha, \beta < 180°$
- along shortest path from x to y,

  paste edge of a doubly covered triangle



  $\Rightarrow$ new vertex @ triangle apex; adds material @ x & y
- continuously vary angles of triangle at x & y from $\emptyset$ to $\alpha/2$ & $\beta/2$ $\Rightarrow$ x & y flatten

$\Rightarrow$ continuous path on manifold of metrics from original metric to metric with one less vertex

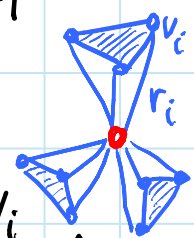
- induct on latter — lost x & y, gain apex
- argue continuity of realizability using Implicit Function Theorem $\Rightarrow$ nonconstructive $\square$

Algorithm for Alexandrov's Theorem: [Bobenko & Izmestiev 2006]

(following Blaschke & Herglotz 1937; Alexandrov 1950; Volkov 1955]

Idea: represent interior of polytope, not just boundary

- add (hypothetical) point p interior to polytope
- triangulate surface with geodesics
- form solid tetrahedron on p & each Δ
- solve for distance r_i from p to vertex v_i



$\Rightarrow$ determines geometry of tetrahedra, hence polytope

Generalized polytope: same combinatorial structure, tetrahedra glued around p, but not necc. in 3D

- consider dihedral angles of edges of tetrahedra ~ view as angle of solid material
- convexity invariant: Σ two dihedral angles incident to edge of surface triangulation $\leq 180°$
- goal: reach real polytope where $\kappa_i = 360° - \Sigma$ dihedral angles around interior edge $(p, v_i) = 0$

Evolution: start at generalized polyhedron $P(0)$

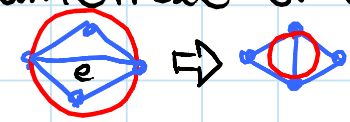
- set $\kappa_i(t) = (1-t)\,\kappa_i(0) \to 0$ as $t \to 1$
- differential equation to evolve r_i's:

$$\frac{d\vec{r}}{dt} = \underbrace{\left(\frac{\partial \vec{\kappa}}{\partial \vec{r}}\right)}_{\text{Jacobian - how } r_i\text{'s affect } \kappa_j\text{'s}}^{-1} \cdot \vec{\varphi}(0)$$

- geodesic triangulation changes (flips) as $t \to 1$
- crucial part of proof: Jacobian has inverse

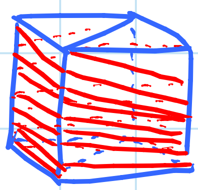
Algorithm for Alexandrov's Theorem: (cont'd)

Starting point: need generalized polyhedron $P(\emptyset)$

1. compute Delaunay geodesic triangulation of surface [Bobenko & Springborn 2005]
   - start with arbitrary geodesic triangulation
   - flipping algorithm: if circumcircle of edge e contains a vertex, flip e 

   - in 2D, $O(n^2)$ flips suffice
   - here, can be arbitrarily many ~ but finite
   - example: can start with "barber pole":

infinitely many geodesic triangulations!



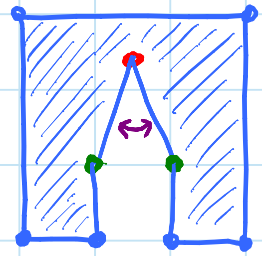
cube with triangulated top & bottom; nasty geodesics on side

2. show that setting all r_i equal & sufficiently large yields desired convexity invariant
   - using Delaunay property

$\boxed{\text{OPEN}}$: bound on running time?

Ungluable polygon: [Demaine, Demaine, Lubiw, O'Rourke 2000]

- no vertex can be glued into red reflex vertex: $< 90°$ free



$\Rightarrow$ "zip" red reflex vertex
$\Rightarrow$ green reflex vertices glued together
$\Rightarrow$ $> 360°$ of material $\square$

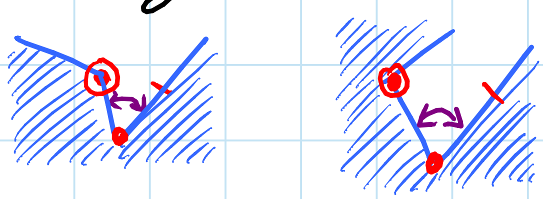
Random polygons are ungluable:

- suppose uniform distribution on angles & edge lengths

$\Rightarrow$ $\approx n/2$ reflex vertices

- gluing in a convex vertex still leaves reflex vertex (angles don't match)
- at some point must zip a reflex vertex
- fails if nearer angle is reflex:

convex $\Rightarrow$ OK



reflex $\Rightarrow$ BAD

- happens with probability $1/2$ for each reflex vertex $\square$

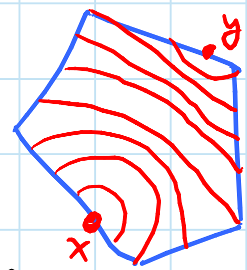
Perimeter halving: every convex polygon has an Alexandrov gluing

- pick any point x on polygon boundary
- glue together two boundary points at distance d from x (measured along boundary), for all $d > \emptyset$
  - both points have $\leq 180°$ of material $\Rightarrow$ convex
- stop at diametrically opposite point y

$\Rightarrow$ gluing two halves (paths) of perimeter from x to y

- x & y also convex (nothing glued)

$\Rightarrow$ Alexandrov $\square$



EXPERIMENT: cut out convex polygon
tape together perimeter halves
see what convex polyhedron you get

Mostly different: uncountably many polyhedra

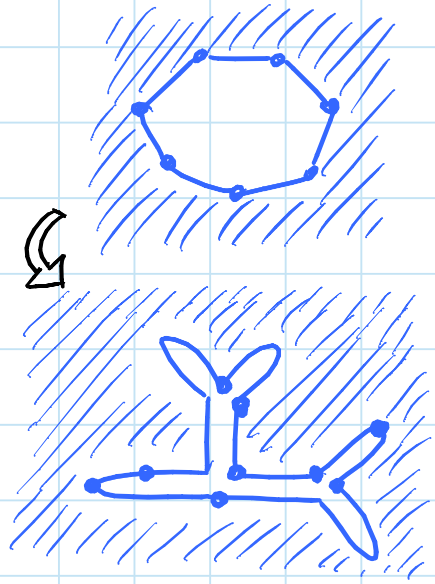
- vary x near vertex v_i, say d along edge $v_i v_{i+1}$
- x & v_i become distinct vertices of shortest-path distance d
- only finitely many vertex-vertex shortest paths for a particular polyhedron
- uncountably many choices for d

$\Rightarrow$ uncountably many polyhedra $\square$

Gluing tree:

- turn polygon "inside-out"
- gluing of that boundary to self forms a cycle around a tree

  gluing tree
- corresponds to cutting tree in unfolding



Properties:

- each leaf is either a zipped vertex or a fold point in middle of edge ($\Rightarrow 180^\circ$)

$\Rightarrow$ at most 4 fold points (720° total curvature)

- if 4 fold points, then these are only leaves

  always induce curvature

  $\Rightarrow$  or 

- at most one nonvertex (middle of edge) glued at ≥ 3-way junction (else $180^\circ \cdot 2 +$ something)

<u>Rolling belt</u> = path in gluing tree whose end points are either fold pts. or convex vx. leaves & along which always $\le 180°$ material on either side

= effectively an embedded convex polygon

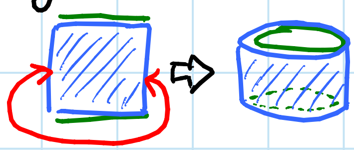
$\Rightarrow$ can perimeter halve arbitrarily = "rolling the belt"

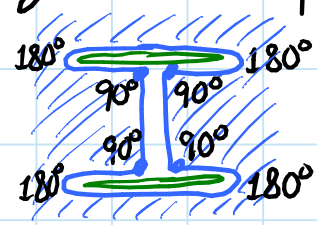
– only way to get infinite gluings

<u>Examples</u>:

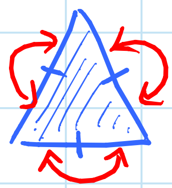
<u>1 rolling belt</u>: perimeter halving of convex polygon

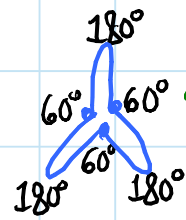
<u>2 rolling belts</u>: cylinder





<u>3 rolling belts</u>:





belt between every pair of leaves

<u>≥ 4 rolling belts</u>: impossible [6.885 Fall 2004 PS5.3]

– must be 4 fold points

$\Rightarrow$ no curvature elsewhere

$\Rightarrow$ rolling belt from one fold point is uniquely determined to some fold point

$\Rightarrow$ same rolling belt from latter fold point

$\Rightarrow$ ≤ 2 rolling belts