6.885	Lecture 11	Oct. 17, 2007

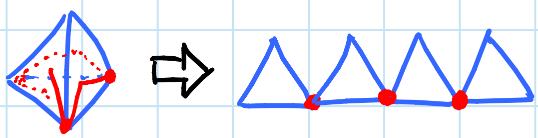
Recall:

	edge unfolding	general unfolding
convex polyhedra	OPEN	ALWAYS
nonconvex polyhedra	NOT ALWAYS	OPEN

Vertex unfolding: [Demaine, Eppstein, Erickson, Hart, O'Rourke 2003]

- different relaxation of edge unfolding
- still cut only along polyhedron edges (all!)
- require connectivity just through vertices (vs. edges)

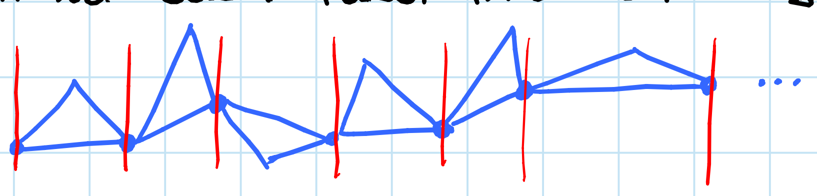
Example:



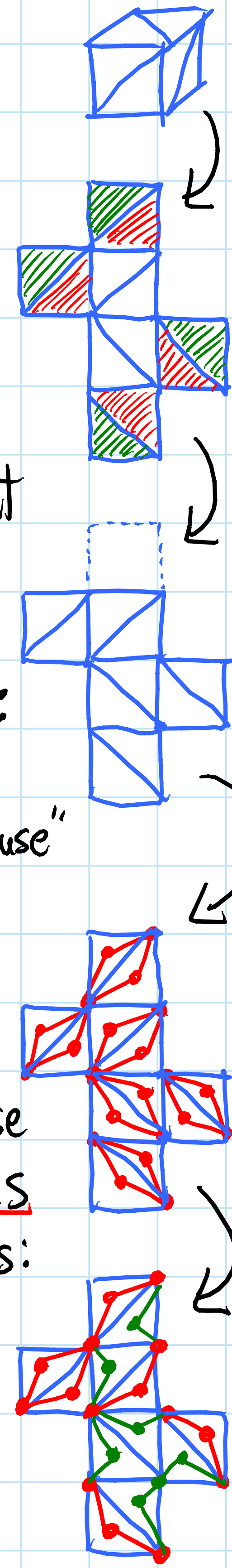
Universality: every connected triangulated manifold has a vertex unfolding, computable in linear time

① construct facet path going from facet to facet along vertex adjacencies, visiting every facet exactly once, not repeating a vertex twice in a row
[next page; also Bartholdi & Goldsman 2004]
↳ but $O(n^2)$ time

② unfold each facet into vertical slab:



⇒ no overlap



Vertex unfolding: (cont'd)

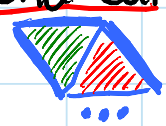
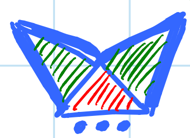
Constructing a facet path for 2D surfaces:

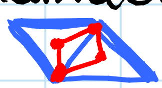
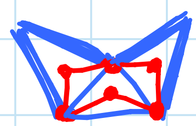
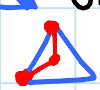
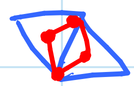
① Cut edges until facets are connected in tree-like fashion
- only removes connections
  - ⇒ only harder to find path
- duplicates vertices but can repeat

→ triangulated "polygon" (may self-overlap)

② Color "ears" = Δs with one adjacent triangle & two boundary edges

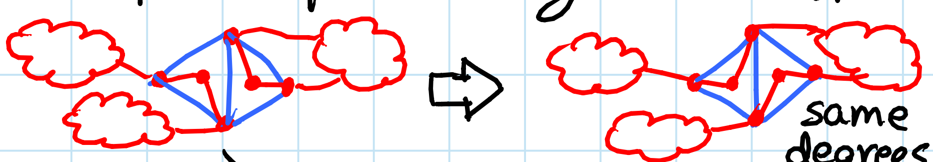
③ Color ears in what remains

④ Remove second ear & 1 or 2 first ear(s):
- either  or 

  "dunce cap" "Mickey Mouse"
- recurse on remainder
- put back with  or 
- base case: nothing or  or 

⇒ every vertex but 2 from base case have even number of connections

⑤ Connect components by local switches:



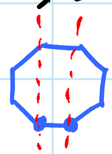
⑥ (Noncrossing) Eulerian path
- get cycle ⇔ not "checkered": 2-color Δs, only red Δs on boundary

Vertex unfolding: (cont'd)

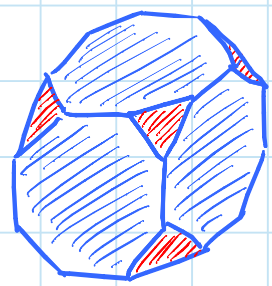
General vertex unfolding is trivial: triangulate

$\boxed{\text{OPEN}}$: does every convex polyhedron have a vertex unfolding?

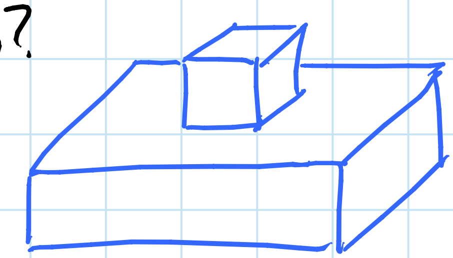
- facet path $\not\Rightarrow$ vertex unfolding:

  no slabs  repeat 3 times

- facet path may not exist:
  - truncated cube has 8 disjoint △s & 6 ⯃s
  - not enough ⯃s to put between △s



$\boxed{\text{OPEN}}$: does every polyhedron with holeless faces have a vertex unfolding?

cf.: 

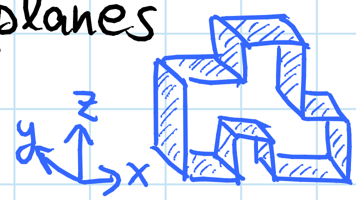
Orthogonal polyhedra: (all faces perpendicular to coord. axis)

generally unfoldable if genus ∅

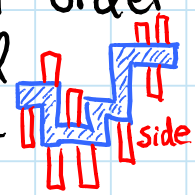
[Damian, Flatland, O'Rourke 2007]

Proof: slice through every vertex with y-plane (xz)

- y-faces encompassed = <u>side faces</u>
- x & z-faces (yz & xy) = <u>band faces</u>
- band faces between two y-planes form a cycle = <u>band</u>

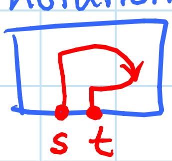


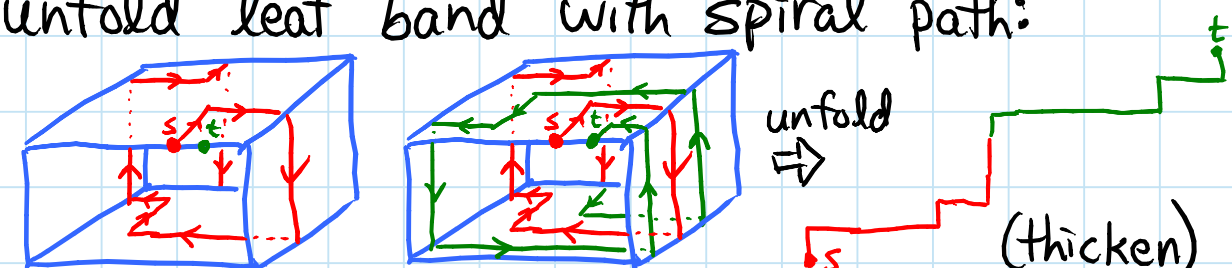
- bands form a tree structure, connecting parent to child with thin side faces
- <u>idea</u>: visit bands in roughly depth-first order
  - unfold to proceed always rightward
  - side faces just attach above/below



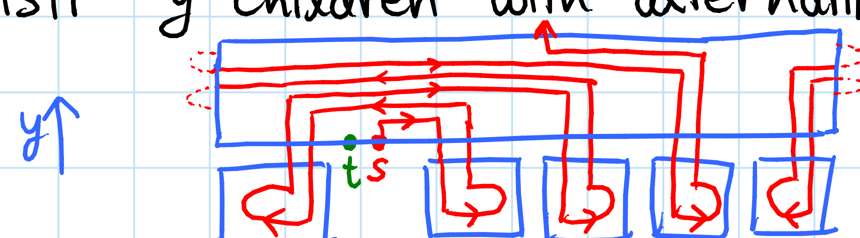
- unfold leaf band with spiral path:

notation:

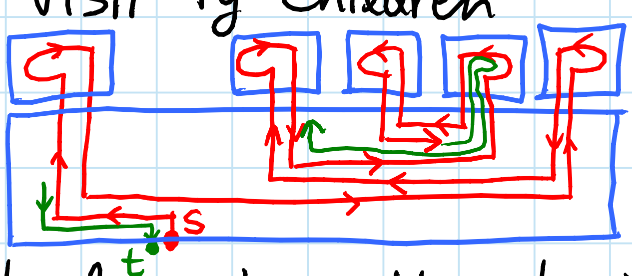




- visit −y children with alternating path:



- similarly visit +y children



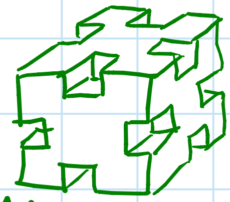
- double-back along entire subtree to return to parent @ <u>t</u>

Orthogonal polyhedra: (cont'd)

Grid = slice through every vertex with x, y, z-planes

Grid unfolding = just cut along grid

$\boxed{\text{OPEN}}$: does every orthogonal polyhedron have a grid unfolding?
— analog of edge unfolding



edge unfolding impossible

Refinement = divide each grid rectangle into $\underbrace{k \times k}_{\text{goal: minimize } k}$

Summary:

	type	$k \times k$	ref
— general		$2^{O(n)} \times 2^{O(n)}$	[DFO 2007] (prev. page)
— general	vertex-unf.	1×1	[DFO 2006]
— Manhattan towers		5×4	[DFO 2005]
↳ connected y=0 base, y-plane slices shrink as y increases			
— orthoterrain (rect. base)		1×1	[O'Rourke 2007]
— orthostacks		2×1	[Biedl et al. 1998]
↳ every y-plane slice is connected			
— orthostacks	vertex-unf.	1×1	[Demaine, Iacono, Langerman 2006]
— orthoconvex orthostacks		1×1	[Damian & Meijer 2004]
↳ y-plane slices are orthogonally convex: x & y slices connected			
— orthotubes		1×1	[Biedl et al. 1998]
↳ unit cubes connected face-to-face in open/closed chain			
— well-separated orthotrees		1×1	[DFMeijerO 2005]
↳ unit cubes connected face-to-face in a tree; ↳ no two branching cubes are adjacent			

Orthogonal polyhedra: (cont'd)

$\boxed{\text{OPEN}}$: $n^{O(1)} \times n^{O(1)}$ refinement in general?

$\boxed{\text{OPEN}}$: $\Omega(n) \times \Omega(n)$ lower bound?

$\boxed{\text{OPEN}}$: power of $O(1) \times O(1)$ refinement?

— e.g. orthotrees

$\boxed{\text{OPEN}}$: power of 1×1 refinement?

— e.g. orthostacks, Manhattan towers

$\boxed{\text{OPEN}}$: orthogonal polyhedra of higher genus?

$\boxed{\text{OPEN}}$: general unfolding of arbitrary polyhedra of genus $\emptyset$?

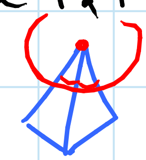
$\boxed{\text{PROJECT}}$: implement $2^{O(n)} \times 2^{O(n)}$ method for unfolding orthogonal polyhedra of genus $\emptyset$

<u>Cauchy's Rigidity Theorem</u>: [Cauchy 1813; Steinitz 1934]

if two <u>convex</u> polyhedra are <u>combinatorially equivalent</u>
same graph/incidence structure

& corresp. faces are congruent, then polyhedra are congruent

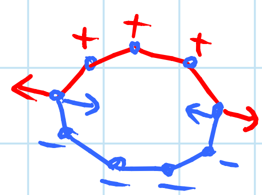
<u>Proof</u> by contradiction: consider counterexample P, P'



- for each vertex pair v, v':
  slice polyhedron with ε-sphere at v, v'
  $\Rightarrow$ spherical polygon, edge lengths = face angles,
  angles = dihedral angles
- faces congruent $\Rightarrow$ edge lengths match in v vs. v'
- P, P' incongruent $\Rightarrow$ angles don't match for <u>some</u> v, v'
- label vertex in v's spherical polygon $+$ if larger angle in v,
  $\emptyset$ if equal angles, and $-$ if smaller angle in v
- can't be all $+$ (& $\emptyset$) or all $-$ (& $\emptyset$) by:
  <u>Cauchy Arm Lemma</u>: opening all angles of a convex open chain increases distance of endpts. (in plane & sphere)

  $\Rightarrow$ edge length would not be preserved

$\Rightarrow \geq 2$ alternations $+ + \cdots + - - \cdots - + + +$



- can't be just 2 by Cauchy Arm Lemma:

$\Rightarrow \geq 4$ alternations $+ - + - +$

$\Rightarrow$ # alternations $\geq 4V$ in subgraph of $+/-$ edges

Proof of Cauchy's Rigidity Theorem: (cont'd)

- +/− labels extend to entire edge of polyhedron
- +/− alternation at vertex corresponds one-to-one +/− alternation in incident face
- $\leq 2k$ alternations in face of $2k$ or $2k+1$ edges

$\Rightarrow 4V \leq \#\text{alternations} \leq 2f_3 + 4f_4 + 4f_5 + 6f_6 + 7f_7 + \cdots$

- $E = \frac{1}{2}(3f_3 + 4f_4 + 5f_5 + \cdots)$ (handshaking)
- $V = 2 + E - F$ Euler's Theorem
  $= 2 + \frac{1}{2}(f_3 + 2f_4 + 3f_5 + \cdots)$

$\Rightarrow 4V = 8 + 2f_3 + 4f_4 + 6f_5 + \cdots \longrightarrow$ contradiction (+8)

$\square$

Uniqueness of folding: [Alexandrov 1941]

- suppose you glue polygon's boundary to itself
- what convex polyhedra can it form?
- every edge will be a shortest path
- draw all shortest paths between vertices (points of nonzero curvature)

$\Rightarrow$ fix combinatorial & facial structure

- Cauchy's Rigidity Theorem $\Rightarrow \leq 1$ convex realization