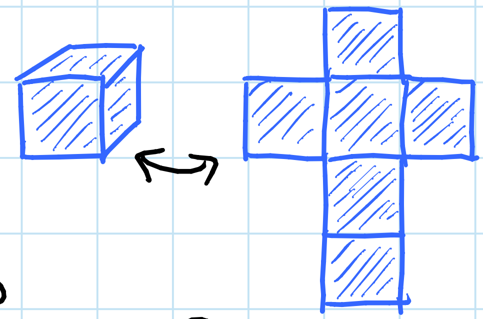
6.885	Lecture 10	Oct. 15, 2007

Part III: Polyhedron (un)folding



Folding: when can a polygon be glued along its boundary to form (exactly) a convex polyhedron?
(only one layer allowed, unlike origami)

Unfolding: when can a polyhedral surface be cut & unfolded into one nonoverlapping planar piece?

Edge unfolding: just cut along polyhedron's edges
General unfolding: can cut interior to faces

Summary:

	edge unfolding	general unfolding
convex polyhedra	OPEN	ALWAYS
nonconvex polyhedra	NOT ALWAYS	OPEN

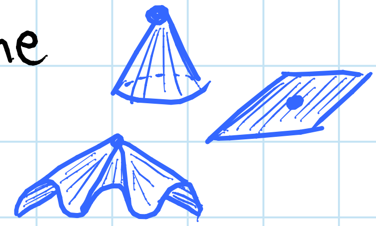
Big questions:

$\boxed{\text{OPEN}}$: does every convex polyhedron have an edge unfolding? [Dürer 1525; Shephard 1975]

$\boxed{\text{OPEN}}$: does every polyhedron without boundary have a general unfolding?
[Bern, Demaine, Eppstein, Kuo 1999]

__Curvature__ of a vertex $= 360° - \sum$ incident face angles

- positive $\Rightarrow$ convex cone
- zero $\Rightarrow$ flat
- negative $\Rightarrow$ saddle



} can be convex
} never convex

__Cutting__ = cuts in a valid unfolding

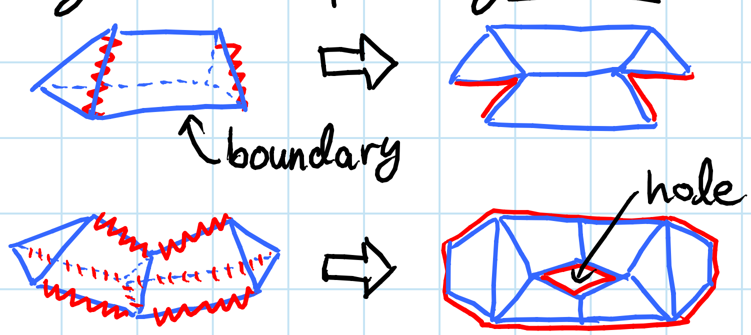
- only zero-curvature vertices can be flattened without cutting or local overlap

$\Rightarrow$ any cutting __spans__ all nonzero-curvature vertices

- indeed, if curvature $< -k \cdot 360°$ then cutting must have degree $> k+1$
- if polyhedron has no handles (sphere/disk, not torus) then cutting has no cycles (else >1 piece)
  $\Rightarrow$ spanning forest
- connected component of cutting makes boundary component of unfolding

$\Rightarrow$ if polyhedron has no boundary or handles & unfolding has no holes then cutting is a __spanning tree__

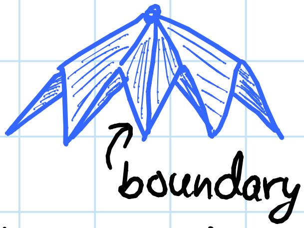
- cf.:



- if polyhedron is convex then cutting is a spanning tree

Trivial bad example: [Bern, Demaine, Eppstein, Kuo 1999]

polyhedron with boundary & just one vertex, of negative curvature



- need ≥ 2 cuts at vertex
- can't stop cutting until we reach the boundary (else could reglue cuts without change)

$\Rightarrow$ disconnect surface

$\Rightarrow$ no general unfolding

Shortest path between two points x & y on polyhedron
- unfolds straight (geodesic)
- doesn't cross itself
- doesn't pass through a positive-curvature vertex

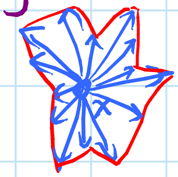
General unfoldings of convex polyhedra:

Star of shortest paths from point x to all other points
- if two shortest paths touch beyond x then either one is a subpath of another or they touch only at their ends ($\Rightarrow$ nonunique shortest path)

Cut locus / ridge tree with respect to point x
= points with nonunique shortest paths from x
- spanning tree of polyhedron
- leaves = the polyhedron vertices

Source unfolding [Sharir & Schorr 1986; Mount 1985; Mitchell, Mount, Papadimitriou 1987]
- cut along the cut locus
- unfold star of shortest paths from x

$\Rightarrow$ star-shaped unfolding: boundary visible from x



Star unfolding [Alexandrov 1948; Aronov & O'Rourke 1992]
- cut along shortest paths from (generic) point x to every polyhedron vertex (star of cuts)
- much harder to prove nonoverlap

General unfoldings of convex polyhedra: (cont'd)

New unfolding: [Itoh, O'Rourke, Vîlcu 2007]
- find noncrossing closed (cyclic) geodesic
- cut all but short segment of it
- cut shortest path from each vertex to geodesic

Extensions:
- neither works for nonconvex polyhedra
- source unfolding works in higher dimensions [Miller & Pak 2003]
- source unfolding can be "continuously bloomed" without intersection [Langerman et al. 2004 unpubl.]
  - $\boxed{\text{OPEN}}$: true of star unfolding?
    all edge/general unfoldings?
- $\boxed{\text{OPEN}}$: other general unfoldings?

Edge-unfolding convex polyhedra: [1525]

- implicitly dates back to Albrecht Dürer's Painter's Manual
- possible for every example we've tried
  - e.g. Archimedean
  - heuristic/exhaustive search: commercial software, JavaView Unfold, JavaGami, Unfold for Blender
  - Schlickenreider [1997] search
- all efficient algorithms we've tried fail
- some simple examples overlap
  e.g. sliver tetrahedron

- random cutting of random convex polyhedron overlaps with probability $\to 1$ as $n \to \infty$
  [Schevon & O'Rourke 1987] hull of rand. pts. on sphere
  - $\boxed{\text{OPEN}}$: prove this empirical observation

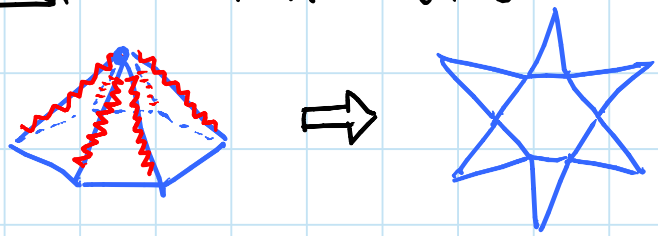
An approach: [Bern, Demaine, Rote, G. Price, ...]

- $\boxed{\text{OPEN}}$: edge-unfold convex terrains
  (project to a plane without intersection)
  $\Rightarrow$ positive equilibrium stress
- $\boxed{\text{OPEN}}$: edge-unfold "almost flat" terrain/polyhedron
  (scale $z \to \varepsilon z$, $\varepsilon \to$ infinitesimal)
  $\Rightarrow$ visible in plane
  - challenging even for prismatoids
    = convex hull of two parallel polygons

Edge-unfolding convex polyhedra: (cont'd)

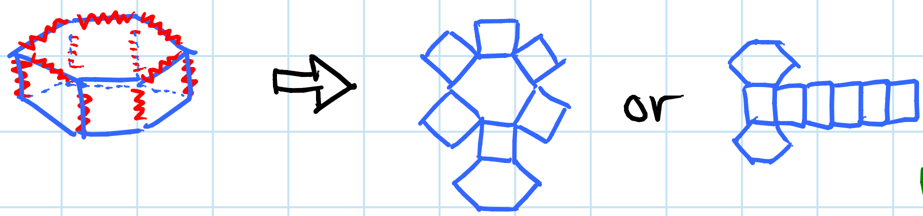
Solved special classes:

- ≤ 6 vertices [DiBiase 1990]
- pyramid = convex hull of convex polygon + point



"Volcano unfolding"

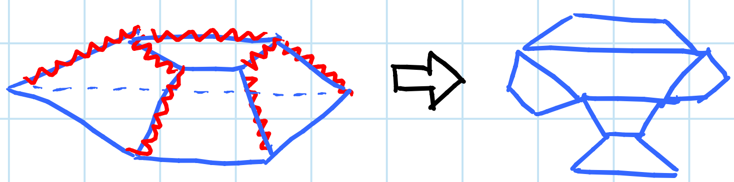
- prism = convex hull of convex polygon + parallel offset



"band unfolding"

- prismoid = convex hull of two parallel convex polygons with matching angles

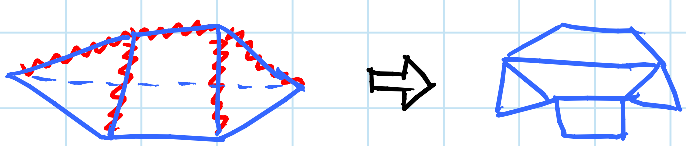
[O'Rourke 2001]



Volcano again

- dome = all faces share edge with single base

[O'Rourke – GFALOP]



Volcano again

- $\boxed{\text{OPEN}}$: prismatoids = convex hull of two parallel convex polygons
  - possible in "smooth" case

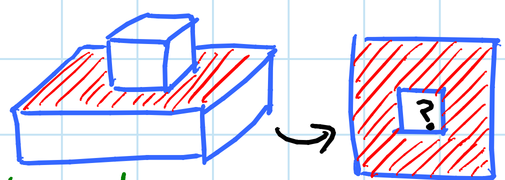
    [Benbernou, Cahn, O'Rourke 2004]

Fewest nets: edge-unfold convex polyhedron into a "small" number of pieces

- want to know whether 1 is possible
- $F = \#$ faces is trivial (cut out each)
- $\frac{2}{3}F$ by pairing together 2/3 of faces [Spriggs 2003]
- $\frac{1}{2}F$ by fancier argument [Dujmović, Morin, Wood 2004]
- better bounds [Pinciu 2007]
- $\boxed{\text{OPEN}}$: $o(F)$ possible?

Edge-unfolding nonconvex polyhedra:

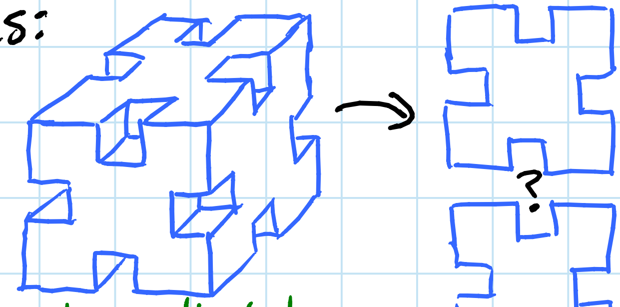
- trivial unfoldable example: insufficient area in donut hole



two boxes

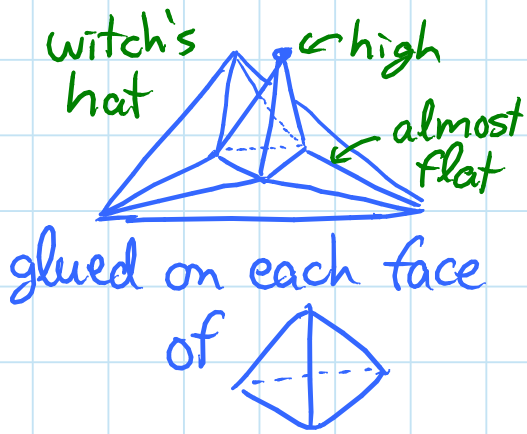
- with all faces ~ disks: can't connect two X's

[Biedl et al. 1999]



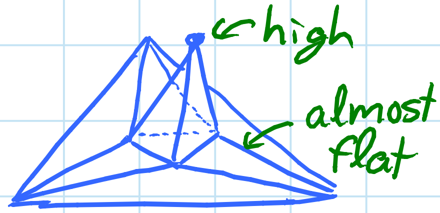
cube with bites

- with all faces triangles
  ⇒ two share only one edge
  ("topologically convex")
  [Bern, Demaine, Eppstein, Kuo, Mantler, Snoeyink 2003]

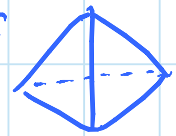


Edge-unfolding nonconvex polyhedra: (cont'd)

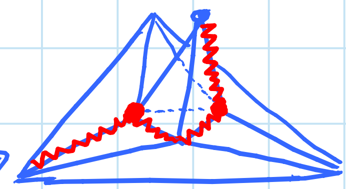
Triangulated ununfoldable example:

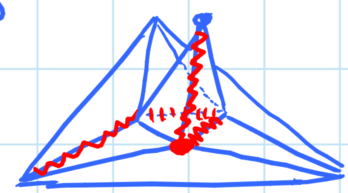


glued on each face of



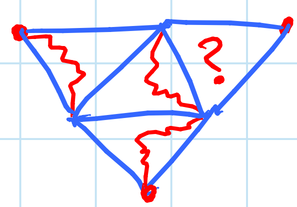
- suppose base vertices of spike have neg. curvature, even without one spike Δ (brim angle $= 300° - \varepsilon$, spike angle $= 90° - \varepsilon \Rightarrow 390° - 2\varepsilon$)
- claim: can't edge-unfold a hat by itself
  - spanning forest has ≥ 2 leaves
  - can't be at negative curvature vertices
  - can't have two on boundary
  - one at peak, one on boundary
  - two possibilities remain
  - both leave all but one spike Δ at a base vertex of spike





$\Rightarrow$ must be a path of cuts between two boundary vertices, interior to hat

- these 4 paths force cycle on 4 vertices



$\Rightarrow$ no one-piece edge unfolding $\square$