6.885	Lecture 9	Oct. 10, 2007

Fold & one cut:

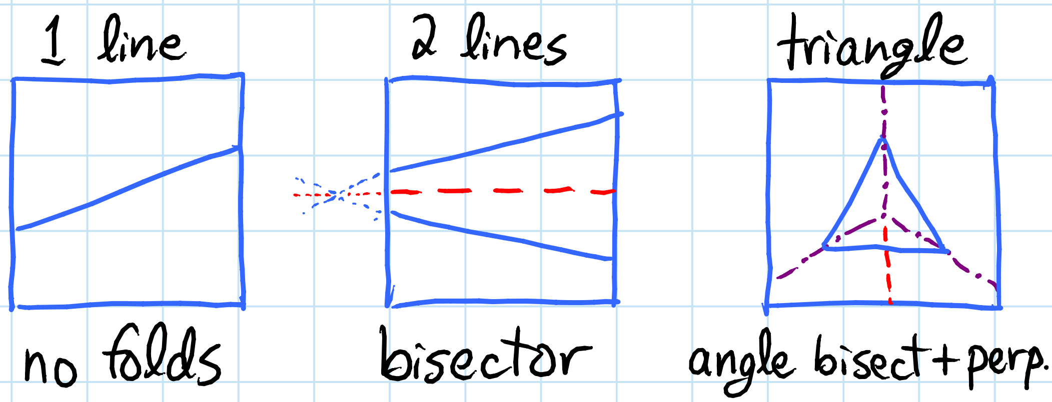
1. fold flat
2. make one complete straight cut
3. unfold

— what shapes/patterns of cuts are possible?

History:
Kan Chu Sen [1721] — Japanese puzzle book
Betsy Ross [1873 story] — ☆ in American flag
Harry Houdini [1922] — ☆ in Paper Magic [ghostw.]
Gerald Loe [1955] — Paper Capers ~ simple folds
Martin Gardner [1960] — Scientific American ~ OPEN

Universality: any set of line segments can be cut
— two methods:
1. straight skeleton [Demaine, Demaine, Lubiw 1998]
   — works almost always; practical
2. disk packing [Bern, Demaine, Eppstein, Hayes 1998]
   — always works; pseudopolynomial; impractical

Warm-ups:

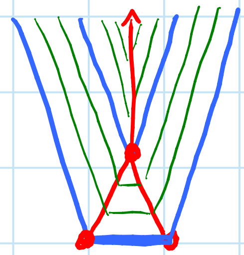


Straight skeleton: [Aichholzer et al. 1995 & 1996]

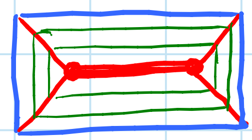
= trajectory of the vertices of the desired cut pattern as we simultaneously shrink each region, keeping edges parallel to the originals & at uniform perpendicular distance

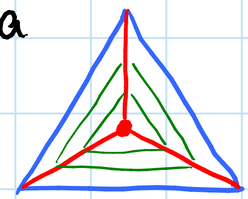
Events during shrinking:

① edge shrinks to $\emptyset$ length
  ⇒ drop it

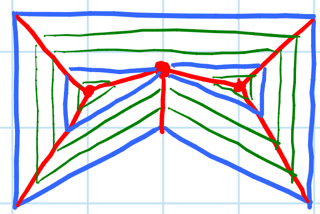


② entire region collapses to $\emptyset$ area
  ⇒ add it all
  & stop shrinking it

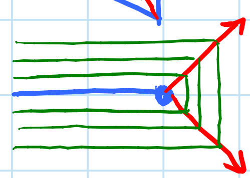




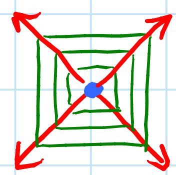
③ face "splits"
  ⇒ recurse in pieces



Degree-1 vertex like end of a rectangle



Degree-$\emptyset$ vertex like a square



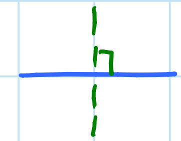
Facts:

- $O(n)$ skeleton vertices, edges, regions
- one-to-one correspondence between cut edges and regions of the straight skeleton
- every skeleton edge is a subsegment of the (angular) bisector of the cut edges corresponding to the two incident skeleton regions ⇒ align

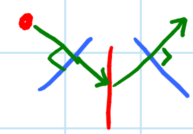
Generic skeleton vertex has degree 3 ⇒ not flat foldable
⇒ need to add some creases...

Perpendiculars: [Demaine, Demaine, Lubiw 1998]

add creases that meet desired cuts
at right angles ⇒ preserve alignment

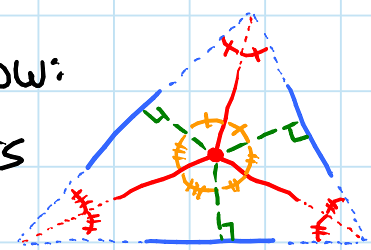


- from each skeleton vertex, try to enter each incident skeleton region with ray perpendicular to corresponding cut edge
- if ray hits another skeleton edge, reflect (⇒ remains perpendicular to corresponding cut edge)

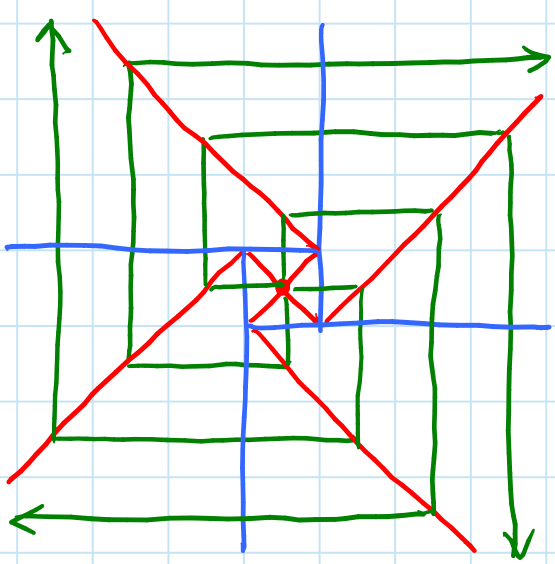


Typical behavior at skeleton vertex now:

- skeleton edges bisect perpendiculars
⇒ Kawasaki condition holds

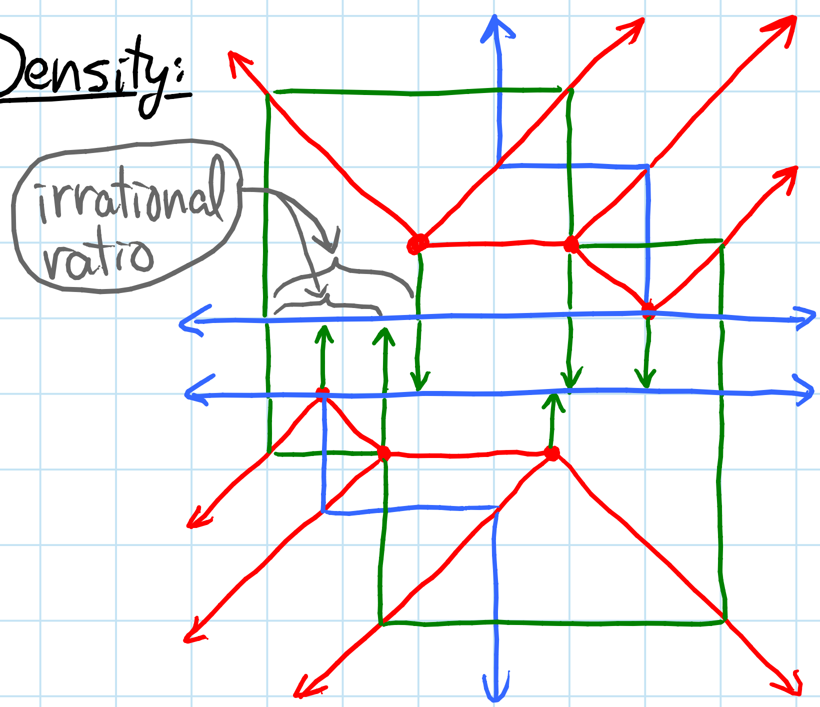


Spiraling:



⇒ infinite creases,
but finite in finite paper

Density:



⇒ creases are dense

CONJECTURE: rare (prob. 0)

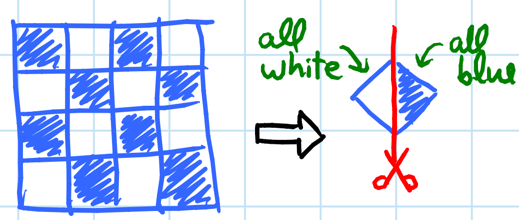
<u>Mountain-valley assignment:</u> (initial)

- skeleton edge <u>mountain</u> if bisects convex angle
  <u>valley</u> if bisects reflex angle
- cut edge <u>valley</u>
- perpendiculars alternate mountain/valley;
  starting to be determined later

<u>Side assignment:</u> specify which cut regions are <u>above</u> or <u>below</u> the cut line

- skeleton edges as above in above regions;
  reversed in below regions
- cut edge <u>valley</u> between two above regions
  <u>mountain</u> between two below regions
  <u>uncreased</u> between one above & one below

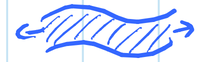
- e.g. 2-regular (nested/disjoint polygons)
  ⇒ natural 2-coloring
  ⇒ all cuts uncreased ("<u>scissor cuts</u>")
- e.g. 4-regular checkerboard

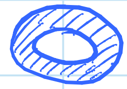


PROJECT: implement crease pattern algorithm (ideally with degeracy tool, M/V assignment, folded state...)

Send me your cool fold & cut examples!

Corridor = region delimited by perpendiculars

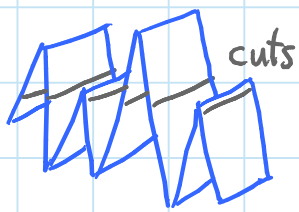
- constant width, measured perpendicularly
- either linear = eventually reach infinity 

  or circular = closed loop 

  ↳ harder to fold (theoretically & practically)
- $\boxed{\text{CONJECTURE}}$: max. degree 2 ⇒ linear corridors only with probability 1

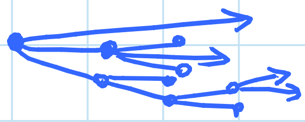
Linear-corridor case: (proof sketch)

- each corridor folds as an accordion
  - alternates mountain/valley
  - aligns cut edges



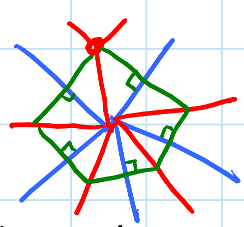
- corridors form a tree structure ≈ projection
  - edge = corridor, length = width
  - vertex = connected component of perpendiculars
- fold tree flat by depth-first search

  ⇒ origami folds flat (argue noncrossing)

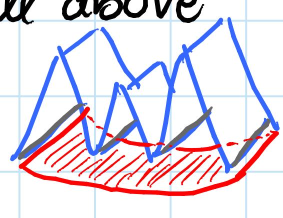


Circular-corridor case:

- trouble: accordion has to wrap around at some edge — reversed: intersect?
- $\boxed{\text{CONJECTURE}}$: with probability 1, circular corridors are normal →



- if normal & side assignment is "all above" then can reverse a cut edge:



Disk-packing method: [Bern, Demaine, Eppstein, Hayes 1998–2006] & O'Rourke

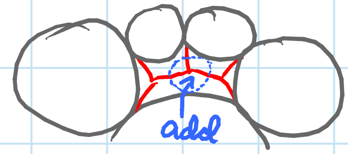
① thicken desired cuts by 2ε
by parallel offset by $\pm\varepsilon$ (ε suff. small)
(just like straight skeleton)

② find a (nonoverlapping) disk packing such that

ⓐ every vertex of offset cuts & paper boundary is the center of a disk
— put small disk at each vertex

ⓑ every edge of … is a union of radii
— pack small disks along each edge

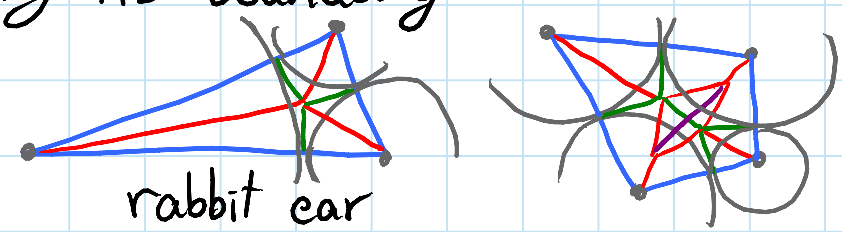
ⓒ every gap between disks has 3 or 4 sides
— repeatedly subdivide gaps:



[Eppstein 1997]

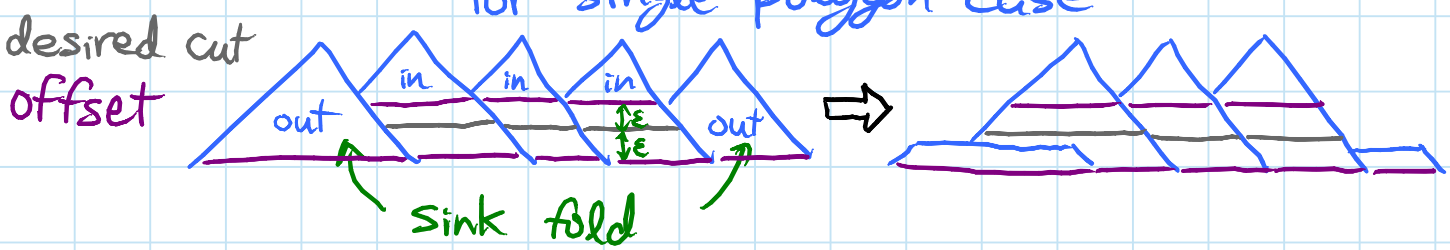
③ dual ⇒ decomposition into triangles & quads.

④ fold each triangle/quad. into molecule aligning its boundary



⑤ glue molecules together ⇒ align all edges
— argue no crossings ~hard part

⑥ sink-fold exterior molecules to height $<\varepsilon$
for single polygon case



Disk-packing method: (cont'd)

- can generalize to arbitrary cut graphs
  (but not arbitrary side assignments)
  - joining & sinking gets messier
- can bound # creases (# disks) in terms of n & integral of "local feature size"
  (distance from x to another boundary point, dx)

$\boxed{\text{OPEN}}$: strongly polynomial bound possible for any solution to fold & cut?

$\boxed{\text{OPEN}}$: what is possible with just simple folds?
($\Rightarrow$ must have a line of symmetry)
[Martin Demaine]

Flattening polyhedra: given polyhedral surface as piece of paper, can it fold flat at all?
[Demaine, Demaine, Lubiw 2000] (without tearing)

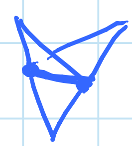
Connection to fold & cut:

	2D fold & cut	3D fold & cut
— paper:	2D region	3D solid
— cuts:	1D segments	2D polygons
— fold:	through 3D	through 4D
— flat:	down to 2D	down to 3D
— so that:	segments on line	polygons on plane

⇒ flattening is boundary of 3D fold & cut

OPEN: d-D fold & cut for $d \geq 3$? e.g. convex polyh.?
OPEN: align all k-D faces, $0 \leq k \leq d$, for $d \geq 2$?
OPEN: flattening based on 3D straight skeleton? [Demaine, Demaine, Lubiw 2000]
— possible for "thin convex prismatoids"

Flat folded state exists [Bern & Hayes 2006; 2007]
— based on disk packing fold & cut [see Ch. 18]



OPEN: arbitrary polyhedral complexes
OPEN: continuous motion?
OPEN: connected configuration space of a polyhedral piece of paper?
— no canonical state — not possible rigidly