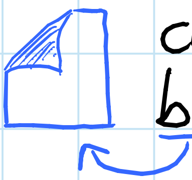
6.885 | Lecture 8 | Oct. 3, 2007

Folding any shape: [Demaine, Demaine, Mitchell 2000]

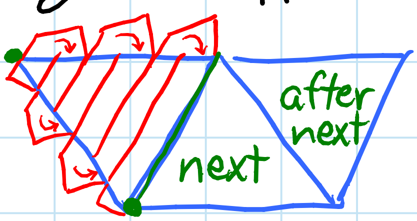
(a.k.a. silhouette [Bern & Hayes 1998] / gift wrapping [Akiyama/Gardner])

Every connected union of polygons in 3D, each with a specified visible color (on each side), can be folded from a sufficiently large piece of bicolor paper of any shape (e.g., square).



Proof: fold paper down to long narrow strip (!)

- triangulate the polygons
- choose a path visiting each triangle at least once
- cover each triangle along the path by zig-zag parallel to next edge, starting at opposite corner:



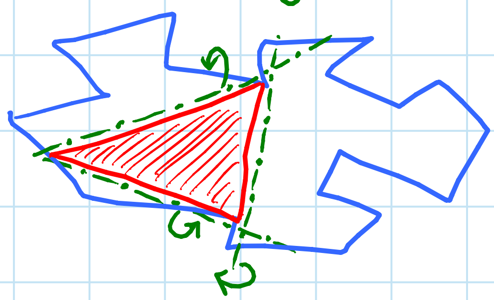
choose parity of zig-zag to arrive at correct corner for next triangle

- turn gadget implements zig-zags & vertex turns:

perpendicular mountain | fold bottom layer | hide excess (many folds)

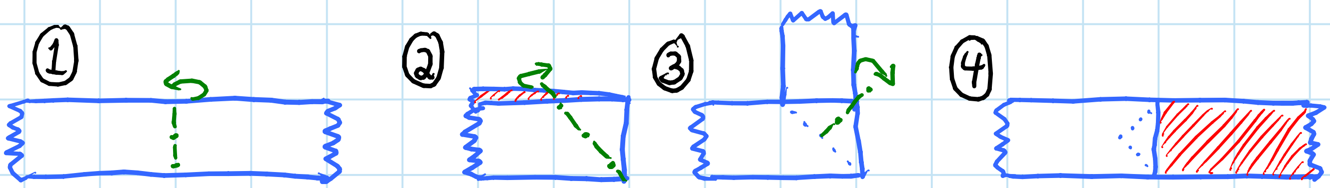
Proof of folding any shape: (cont'd)

- hide excess paper underneath each triangle:
  (more generally, can hide under any convex polygon)



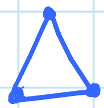
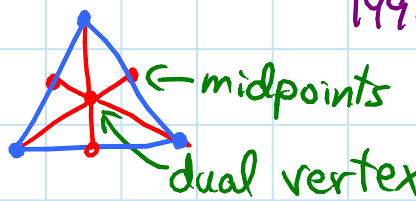
repeatedly mountain fold along lines extending desired edges

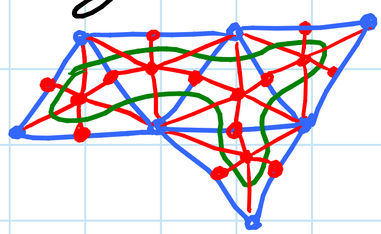
- color-reversal gadget along transition between triangles of opposite colors:



□

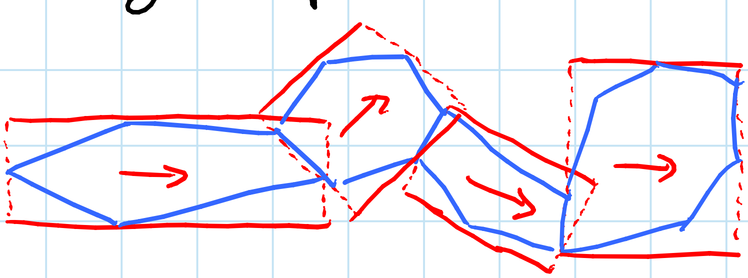
<u>Pseudo-efficiency</u>: if allowed to start with any rectangle of paper, then can achieve
area(paper) = area(surface) + ε for any $\varepsilon > 0$

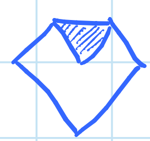
<u>Proof</u>: construct <u>Hamiltonian refinement</u> [Arkin et al. 1996] of triangulation:
- cut each  into 

- walk around spanning tree of original dual: 
- now visit each triangle exactly once
- wastage from turns $\to 0$ with strip width. $\square$

$\boxed{\text{OPEN}}$: pseudopolynomial upper bound? lower bound?

<u>Seam placement</u>: can place seams (visible creases/paper boundary) as desired, provided regions between seams are <u>convex</u>
- <u>idea</u>: vary strip width, use hide gadget



$\boxed{\text{OPEN}}$: what seam placements are possible? 

$\boxed{\text{OPEN}}$: can a given polygon of paper fold into a given target polygon? likely NP-hard

$\boxed{\text{OPEN}}$: what is the smallest <u>square</u> that can fold into a given shape? NP-hard?

<u>Cube wrapping</u>: [Catalano-Johnson & Loeb 2001]

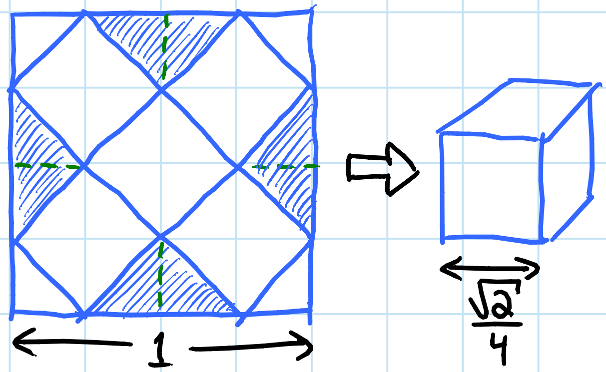
- consider 1×1 square
- in $x \times x \times x$ cube, every point has an antipodal point $\geq 2x$ away

$\Rightarrow$ center of square must be $\geq 2x$ away from corner (points only get closer by folding)

$\Rightarrow$ opposite corners have distance $\geq 4x$

$\Rightarrow$ side length $\geq 2\sqrt{2}\,x$

$\Rightarrow x \leq 1/2\sqrt{2} = \sqrt{2}/4$ & this is possible



$\boxed{\text{OPEN}}$: optimal square $\to$ regular tetrahedron?

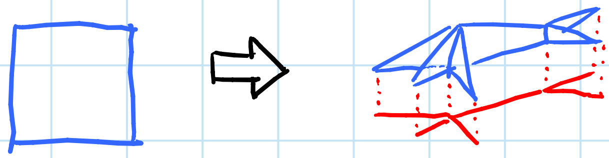
$\boxed{\text{OPEN}}$: $x \times y$ rectangle $\to$ largest cube?

- strip method efficient as $x/y \to \infty$

$\boxed{\text{OPEN}}$: optimal square $\to$ unit $k \times k$ checkerboard

- <u>conjecture</u>: $k/2$ for even $k \geq 4$
- no real lower bounds
- seamless?

Tree method: [Lang 1994–2003; Lang & Demaine 2004–]
algorithm to find folding of smallest square into "uniaxial" origami base whose projection is a desired metric tree



But: – optimization is difficult: exponential time, as hard as disk packing, but good heuristics
– non-self-intersection is only conjectured (we're working on it)

Uniaxial base:
1. in $z \geq 0$ half-space
2. intersection with $z=0$ plane = projection onto that plane
3. partition of faces into flaps, each projecting to a line segment ($\Rightarrow$ all faces vertical)
4. hinge crease shared by two flaps projects to a point: common endpoint of flap projections
5. graph of flap projections as edges, connected when flaps share a hinge crease, is a tree (shadow tree). Hinge creases projecting to a vertex form a hinge
6. only one point of paper folds to each leaf

Tree method: (cont'd)

Key lemma: in any uniaxial base from convex paper,
distance between two points on shadow tree
$\le$ distance between corresponding points on paper

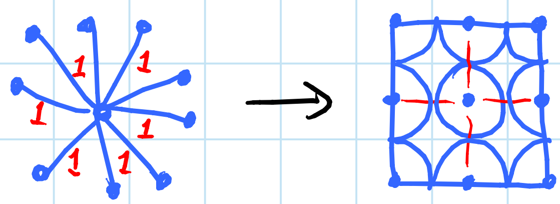
Proof: latter = length of line segment on paper
- folds to path in uniaxial
- projects to shorter path on shadow tree
- shortest path in tree is only shorter $\square$

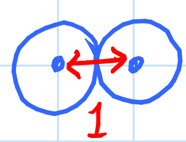
Scale optimization: focus on shadow leaves i
& placement as points p_i on paper:

$$\begin{cases} \text{maximize } \lambda \\ \text{subject to } \underbrace{d(p_i, p_j)}_{\text{distance on paper}} \ge \lambda \cdot \underbrace{d(i,j)}_{\text{fixed distance in tree}} \text{ for leaves } i, j \end{cases}$$

— quadratic constraint

Example:





$\Rightarrow$ radius $= \frac{1}{2}$

star → disk packing, centers in square

$\Rightarrow$ with $n \times n$ piece of paper, get $(n+1)^2$ arms in star; can flatten to perimeter $\Theta(n^2)$

→ MARGULIS NAPKIN PROBLEM

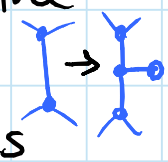
[Lang 2003]

Tree method (cont'd)

Active path = path between two shadow leaves of length = distance in piece of paper

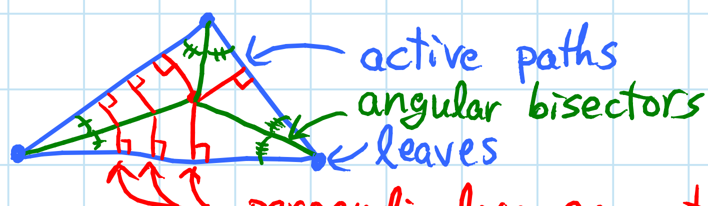
- never cross each other [GFALOP Lem. 16.4.2]

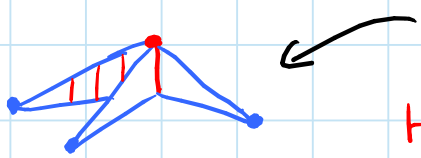
Triangulation: can add artificial leaf edges to the shadow tree to make the active paths partition the piece of paper into triangles (without changing scale factor) [GFALOP Lem. 16.6.2]



- later these leaf edges can be "folded away"
- some triangle edges are paper boundary, not active

Rabbit-ear molecule:

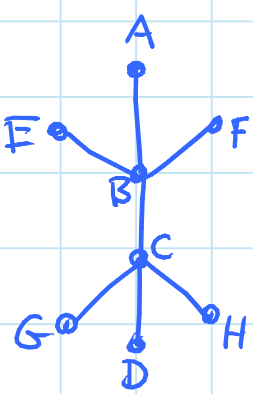


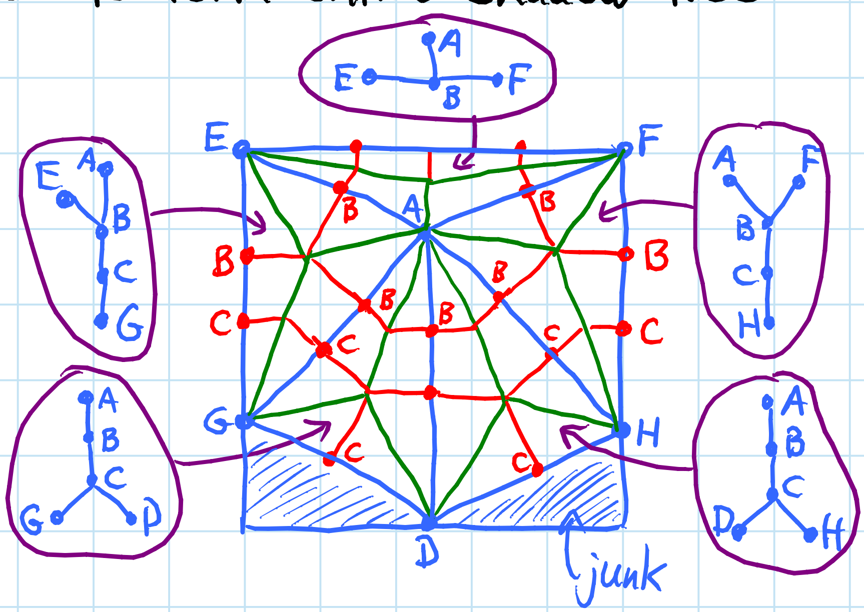


HINGES = perpendiculars emanating from tree vertices along path

- put them together to form entire shadow tree

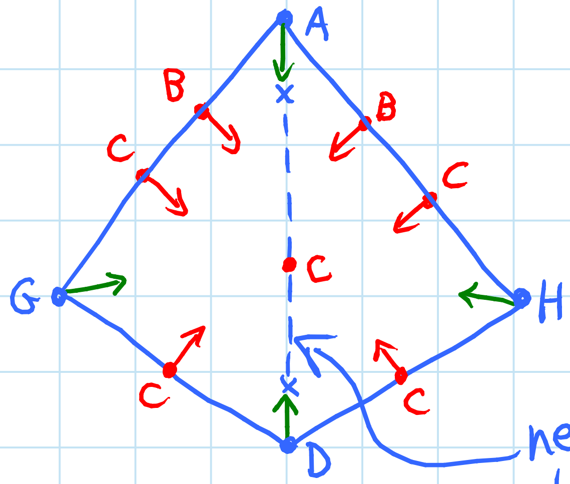
Example:





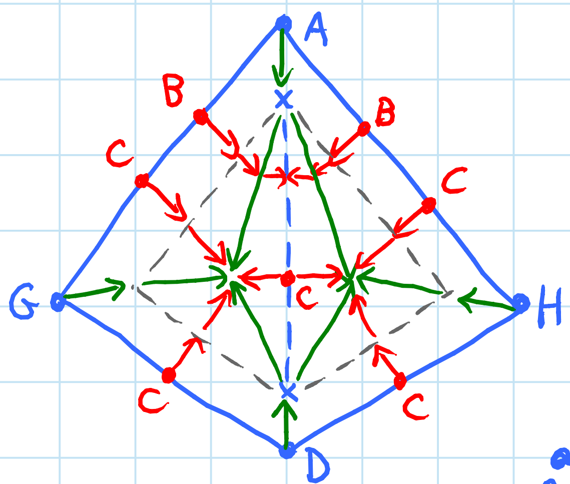
More practically:

- use convex decomposition instead of triangulation (in practice by letting tree edge lengths vary a bit)
- Lang Universal Molecule folds convex polygon



angular bisectors
perpendiculars

GUSSET:
new active path at some point



angular bisectors
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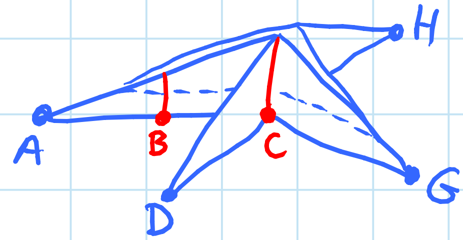

2 kinds of events:

1. gusset: new active path → split shrunken polygon
2. two vertices meeting → continue along new angular bisector