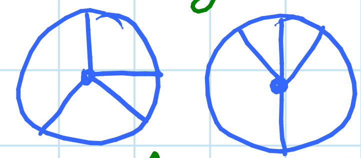
6.885	Lecture 7	Oct. 1, 2007

Single-vertex crease pattern (without loss of generality)

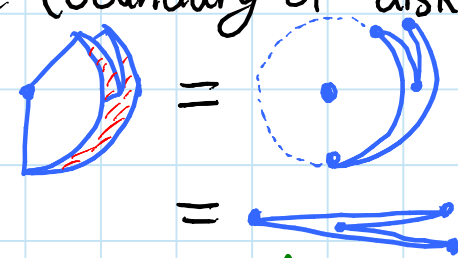
= disk of paper,
creases emanate from center



Idea: capture local foldability around a vertex

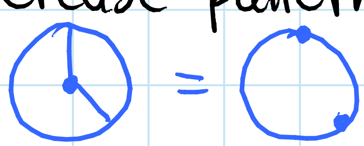
= circular sequence of angles $\theta_1, \theta_2, \dots, \theta_n$
- normally, $\theta_1 + \theta_2 + \dots + \theta_n = 360°$
- allow other sums, especially $\leq 360°$ (convex cone)
  in particular for induction

Flat folding = folding of 1D circle (boundary of disk) on the circle
= folding of 1D circle onto line

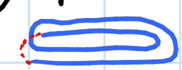


(assuming convex cone & at least one fold
$\Rightarrow$ can't reach all the way around circle)

Differences from 1D (segment) flat folding:
- not all crease patterns are flat foldable:



- equilateral $\not\Rightarrow$ all mountain-valley patterns possible
  e.g. all valleys



- mingling $\not\Rightarrow$ existence of crimp:
  could have ⋯ ( ] ( ] ( ] ⋯ circularly

Characterization of flat-foldable single-vertex crease pat.:

[Kawasaki 1989; Justin 1989; Hull 1994]

$\theta_1, \theta_2, \ldots, \theta_n$ is flat foldable convex cone

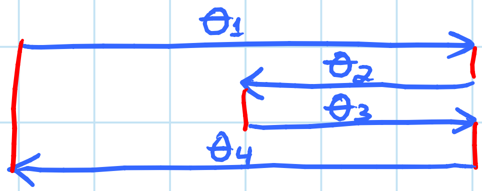
$\iff \theta_1 + \theta_3 + \cdots + \theta_{n-1} = \theta_2 + \theta_4 + \cdots + \theta_n$ (& n even)

$= 180^\circ$ for flat paper

Proof:

($\Rightarrow$) – angles θ_i measure travel on circle/line

– creases switch direction of travel



$\Rightarrow$ n must be even (cycle of switches)

& total motion $= \pm(\theta_1 - \theta_2 + \theta_3 - \theta_4 + \cdots + \theta_{n-1} - \theta_n)$

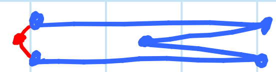
– total motion $= 0$ to end where we started

(assuming convex cone & at least one fold — else $\equiv 0 \pmod{360^\circ}$))

$\Rightarrow$ alternating sum of angles $= 0$

($\Leftarrow$) – cut at an "extreme" crease (e.g., leftmost)

$\Rightarrow$ 1D segment crease pattern

– fold flat e.g. accordion 

– two ends corresponding to cut crease are aligned because total motion $= 0$ & can join because extreme $\square$

Nonconvex cone: $\theta_1 - \theta_2 + \cdots = 0$ or $\pm 360^\circ$ [Demaine & O'Rourke 2007]

Flat-foldable single-vertex mountain-valley patterns

Count: #mountains − #valleys = ±2 in convex cone [Maekawa; Justin 1986]

Proof: measure total turn angle = 180° − interior angle
(>0 for convex, <0 for reflex vertices)

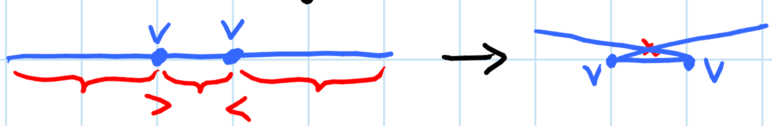
- mountain turns +180°, valley turns −180°
- small turn caused by circle, but cancels out assuming convex cone ⇒ can't reach around
- no crossing ⇒ total turn angle = ±360°

⇒ 180°·#mountains − 180°·#valleys = ±360°

⇒ #mountains − #valleys = ±2. □

Nonconvex cones: if $\theta_1 - \theta_2 + \cdots = \pm 360°$, $\#M - \#V = 0$

Generic case: strict local minimum angle is surrounded by one mountain & one valley



[Kawasaki 1989; Justin 1984]

⇒ Can immediately crimp any such angle
- preserves flat foldability as before for 1D segments:

Remaining case: equal angles

Characterization of flat-foldable single-vertex mountain-valley pattern:

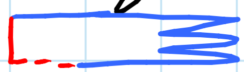
[Hull 2001 & 2003; Demaine & O'Rourke 2007]

Local counts: Among k equal angles surrounded by strictly larger angles (e.g. globally smallest angle),

$$\#\text{ mountains} - \#\text{ valleys} = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \pm 1 & \text{if } k \text{ is even} \end{cases}$$

Proof: build cone from k equal angles & larger angles

- if k even then extend one larger angle to match the other



- if k odd then add new angle of Σ larger angles $-$ equal angle

$\Rightarrow$ flat folding of cone with same M-V assign.

& $\theta_1 - \theta_2 + \cdots + \theta_{n-1} + \theta_n = 0$

- Maekawa's Theorem $\Rightarrow$ # mountains $-$ # valleys $= \pm 2$

  (cone might be nonconvex but still $\theta_1 - \theta_2 + \cdots = 0$)

- if k even then one new crease

  $\Rightarrow 180°$ turn + new crease $\Rightarrow \#M - \#V = \pm 1$

- if k odd then two new creases (same dir.)

  $\Rightarrow 0°$ turn + 2 new creases $\Rightarrow \#M - \#V = 0$. □

$\Rightarrow$ there is at least one crimp among these creases

- applies unless all angles are equal

  $\Rightarrow$ crimp exists by # mountains $-$ # valleys $= \pm 2$ (or 0)

  - unless just 2 creases $\Rightarrow$ same direction

    (or opposite direction if $\theta_1 - \theta_2 = \pm 360°$)

- linear-time algorithm (maintain crimps)

Combinatorics of single-vertex mountain-valley patterns:

linear-time algorithm to count [Hull 2003]

- smallest in generic case $\Rightarrow 2^{\underbrace{n/2}_{\text{number of crimps}}}$

  choices per crimp

- largest in equal-angle case $\Rightarrow 2\binom{n}{\underbrace{n/2-1}_{\text{Ms \& Vs}}}$

  #M − #V = +2 or −2

Continuous single-vertex foldability: [Streinu & Whiteley 2001]

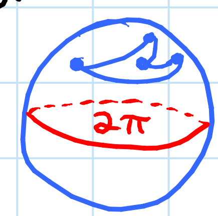
every folded state of a single-vertex crease pattern can be folded continuously <u>*without extra creases*</u>

Spherical Carpenter's Rule Theorem:

closed chain of total length $\leq 2\pi$ on unit sphere has a connected configuration space

- proof based on projective invariance of infinitesimal rigidity
- length $\leq 2\pi \Rightarrow$ lie in hemisphere

  $\Rightarrow$ can project to plane



- length $> 2\pi \Rightarrow$ no convex configuration

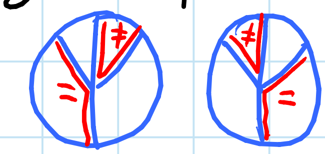
Touching case (e.g. flat folding) handled by recent self-touching Carpenter's Rule Theorem [Abbott, Demaine, Gassend 2007]

Local foldability: [Bern & Hayes 1996]

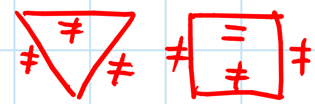
linear-time algorithm finds a consistent mountain-valley assignment (if possible) such that each vertex locally folds flat

Proof: all possible mountain-valley assignments of a single vertex generated by crimps

- crimped pair forced unequal
- final pair forced equal



- cycles can have parity issue:



- Pairing unique in generic case
- if equal angle next to crimped angle then can interchange order of crimps
- merge if interchange decreases # paths/cycles of =/≠ constraints
- merges only fix parity problems
- cook until done $\square$

PROJECT: implement local foldability test converting crease pattern → M-V pattern

OPEN: minimum number of added creases to make given crease pattern [locally] flat foldable

- with or without mountain-valley assignment
- always possible via disk-packing fold & cut

Hardness of global flat foldability: [Bern & Hayes 1996]

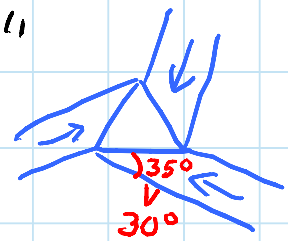
① deciding flat foldability of given crease pattern is strongly NP-hard

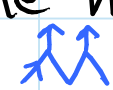
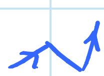
② constructing valid λ ordering for given flat-foldable mountain-valley pattern is strongly NP-hard

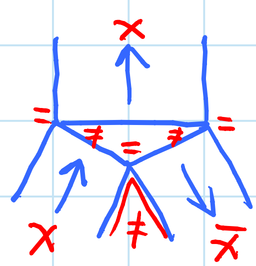
Proof: (①) reduce from <u>all-positive not-all-equal 3-satisfiability</u>: given triples (x_i, x_j, x_k), is there a Boolean assignment to $x_1, x_2, \dots, x_n$ such that no triple is all-true or all-false?

<u>Wire</u> = "pleat" = two close parallel creases
false $\Leftrightarrow$ left <u>mountain</u>

<u>NAE clause</u> = triangular "overtwist"
– can't all fold same way
(twist is borderline)



<u>Reflector</u> splits wire x into two copies, one negated
⇒ split gadget 
& turn gadget  (with noise)
⇒ can connect variable wires to desired clauses



Also need crossover gadgets. □