6.885	Lecture 6	Sept. 26, 2007

Part II: Paper folding

High level:

- usually 2D paper ~ but 1D & $3D^+$ cool too
- no stretching/shrinking/tearing
- no crossing (but touching is O.K.)
- difference from linkages:
  creases can go anywhere (fibers, not bars)

Main topics:

① definitions/technicalities
- surprisingly intricate
- connectivity of configuration space

② foldability
- crease patterns & when they fold

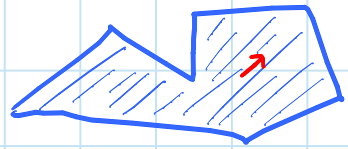
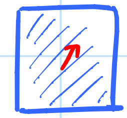
③ design
- given specific geometric goal (shape/property/etc.)
- find crease pattern & folding

④ geometric construction
- analog of straight edge & compass (used all the time by origamists)
- but more powerful!

Definitions: [Demaine, Devadoss, Mitchell, O'Rourke 2004; Lang 2004]

Piece of paper = "nice" orientable manifold (↗ see Chap. 11)

usually: – 1D line segment 

– 2D polygon  

Distinguish "top side"

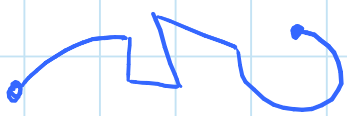
Folded-state geometry of d-dimensional paper P

= isometric mapping into $(d+1)$-dim. space $f: P \to \mathbb{R}^{d+1}$

↳ preserve intrinsic distances (no stretch/shrink)

Free folded state = one-to-one folded-state geometry

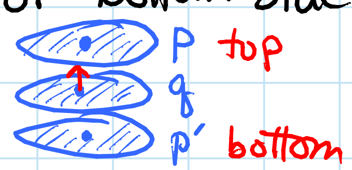
↳ no touching ⇒ no crossing



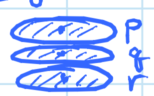
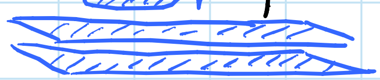
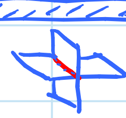
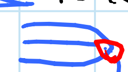
Folded-state order

= for every two distinct noncrease (↗ smooth) points p, q of paper collocated by the geometry — $f(p) = f(q)$ — specify whether p is on top or bottom side of q:

$\lambda(p, q) = +1$ or -1



satisfying:

1. antisymmetry:  $\lambda(p,q)$ vs. $\lambda(q,p)$
2. transitivity:  p above q above r
3. consistency:  touching regions
4. noncrossing: no  or 

Folded state = geometry + order

Definitions: (cont'd)

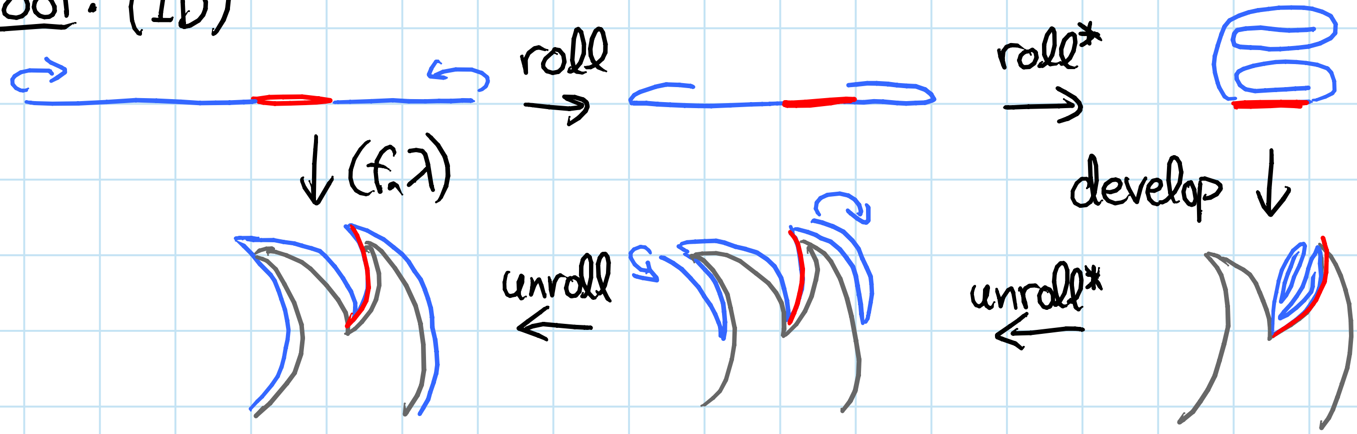
$\boxed{\text{OPEN}}$: simpler definition using infinitesimals?

→ see Chap. 11 for continuity def.

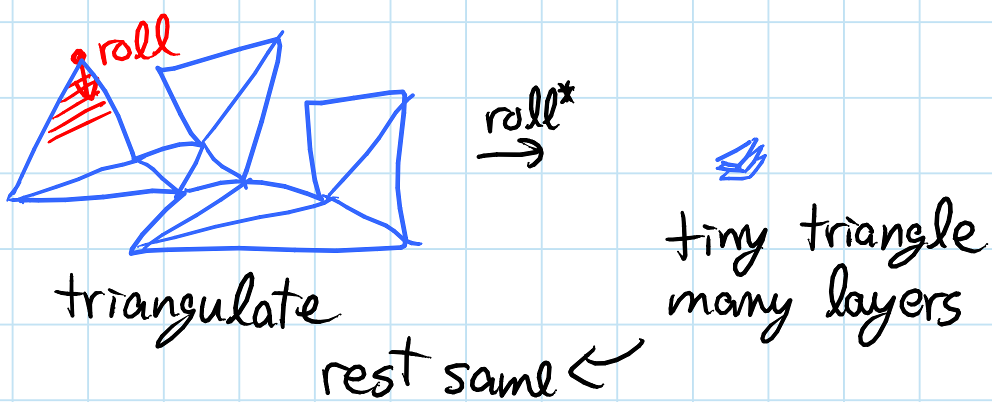
Folding motion = continuum of folded states
$m: t \in [0,1] \mapsto (f_t, \lambda_t)$

Connectivity of configuration space [Demaine, Devadoss, Mitchell, O'Rourke 2004]
any polygonal piece of paper P
has a folding motion into any desired folded state (f, λ)

Proof: (1D)



(2D)



$\square$

$\boxed{\text{OPEN}}$: finite # creases, if folded state is free?

$\boxed{\text{OPEN}}$: polygon with holes? (unknotted) polyhedral paper?

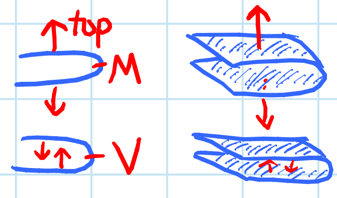
Origami terminology: (d-dimensional paper)

Crease pattern = set of crease points on paper
→ (d−1)-dimensional

Flat folding = folded state lying in d dimensions
— call its crease pattern *flat foldable*
(must use all creases)

Mountain crease = bottom sides touch

Valley crease = top sides touch



Mountain-valley assignment
= which creases in crease pattern are mountain/valley

origami notation: −·−·−· / − − − −

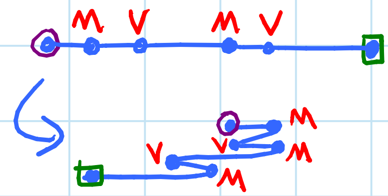
Mountain-valley pattern = crease pattern
+ mountain-valley assignment

Example: crumple up a piece of paper

1D flat folding: [Arkin, Bender, Demaine, Demaine, Mitchell, Sethia, Skiena 2004]

All crease patterns are flat foldable:

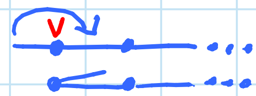
zig-zag / accordion fold



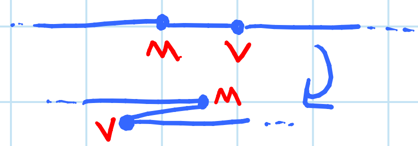
Not all mountain-valley patterns:

Two folding operations:

1. end fold if end length ≤ neighbor

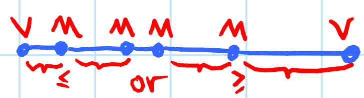


2. crimp two consecutive creases
   if length between ≤ both neighbors
   & one mountain, one valley



Characterization:

mountain-valley pattern is flat foldable
⟺ there's a sequence of crimps & end folds
⟺ mingling: for any maximal sequence of M's or V's,
adjacent V or M or end on at least one side
is nearer than adjacent M or V:

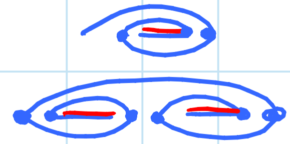


$\boxed{\text{OPEN}}$: proof without mingling?

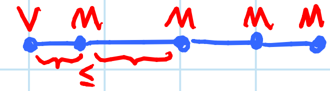
1D flat folding (cont'd)

Proof of characterization:

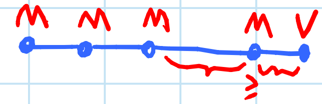
- flat foldable ⇒ mingling:
  - sequence of M's or V's form a spiral (or double spiral)
  - at least one end must be short



- mingling ⇒ end fold or crimp possible
  - for each maximal sequence of M's or V's
    write (if "left mingling"
    [otherwise



    & write) if "right mingling"
    ] otherwise



  ⇒ (] or [) or ()
  -)(⇒ crimp
  - leading (/ trailing) ⇒ end fold
  - if neither: [)[)⋯[)(](]⋯(]

    crash

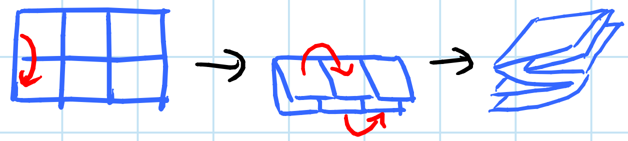
- crimp/end fold preserves flat foldability
  - take flat folding
  - move some layers out of crimp

  ⇒ could start with crimp

- induct ⇒ sequence of crimps & end folds
  ⇒ flat foldable again. □

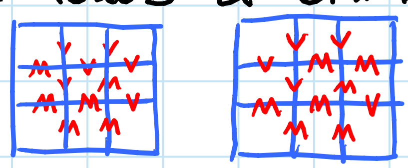
2D map folding: [Arkin et al 2004]

↳ rectangular paper with axis-parallel creases

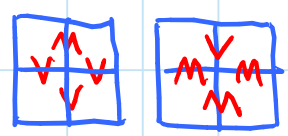
- again every crease pattern is flat foldable:
  zig-zag in x then y 

$\boxed{\text{OPEN}}$: characterize flat-foldable mountain-valley patterns — even $2 \times n$! [Edmonds 1997]

Simple folds: fold by 180° along one line at once

- in 1D, simple folds are omnipotent:
  can simulate end folds & crimps
- in 2D, it matters: 

Characterization of simple foldability of maps:

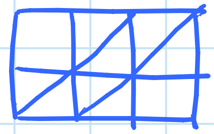
- if simply foldable, must be a uniform horizontal/vertical line
  ↳ all M or all V
- crossing vert./horiz. lines must switch M↔V here:
  - local 2×2 patterns:  & rotations

⇒ all uniform horizontal lines must be folded before any vertical lines become uniform, etc.

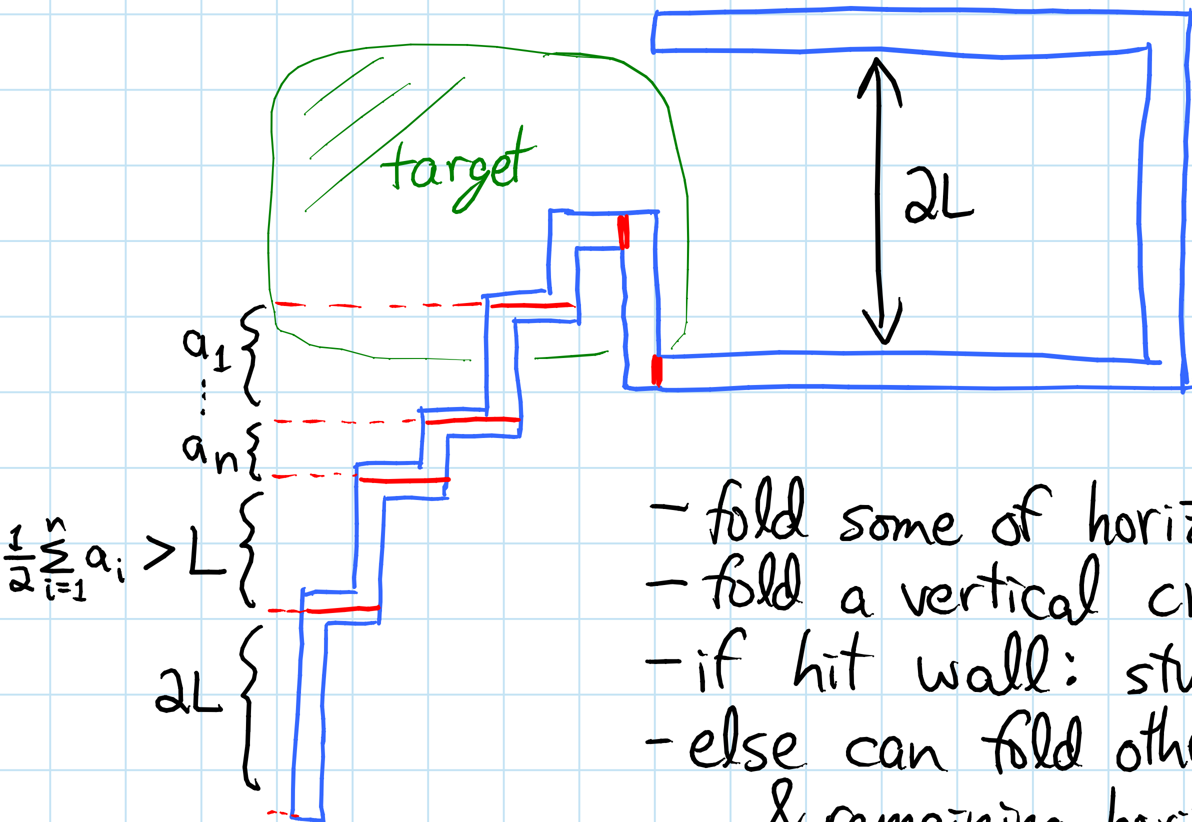
⇒ sequence of 1D problems on current uniform lines

- linear-time algorithm — O(mn) for m×n map — by maintaining uniformity for each line, crimpability & end foldability on lengths

2D map folding (cont'd)

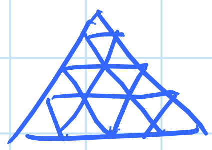
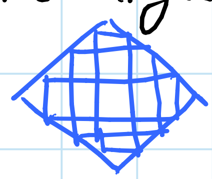
NP-hard if we add 45° diagonal creases 
or allow orthogonal paper

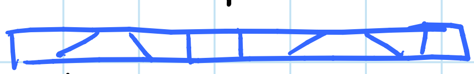
- reduction from Partition: (weakly NP-complete)
  given n integers, can you partition them
  into two groups of equal sum



- fold some of horiz. creases
- fold a vertical crease
- if hit wall: stuck
- else can fold other vertical
  & remaining horiz. creases

$\boxed{\text{OPEN}}$: orthogonal creases in ortho. convex polygon?
in convex polygon? in nonaligned rectangle?

$\boxed{\text{OPEN}}$: triangular maps?  



$\boxed{\text{OPEN}}$: strip with no crossing creases? [Martin Demaine]
(45°, 30/60°, arbitrary...)