6.885	Lecture 5	Sept. 19, 2007

Locked linkages: recall

	chains	trees	
2D	never locked L4 ✓	can lock	TODAY
3D	can lock	can lock	
4D+	never locked	never locked	

Algorithms for unfolding 2D chains

① ordinary differential equation given by (canonical) expansive infinitesimal motion

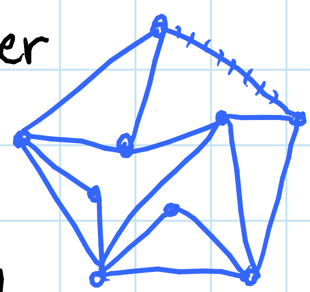
[Connelly, Demaine, Rote 2002]

- strictly expansive
- one step in poly. time: convex program
- many steps; inaccurate (without projection)
  $\boxed{\text{OPEN}}$: how many? pseudopolynomial?

② pointed pseudotriangulations [Streinu 2000]

↳ maximal edge set on given points with >180° angle at every vertex

- expansive
- $n^{O(1)}$ steps
- one step follows 1 D.O.F. linkage → delete edge of convex hull
  - best algorithm is exponential
  - $\boxed{\text{OPEN}}$: are pseudotriangulations easier than general 2D linkages? (e.g. they are noncrossing)
- $\boxed{\text{PROJECT}}$: implement this algorithm



Algorithms for unfolding 2D chains: (cont'd)

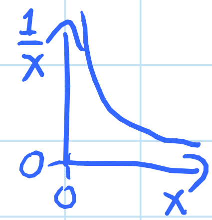
③ energy [Cantarella, Demaine, Iben, O'Brien 2004]

- not expansive
- one step is $O(n^2)$ & exact on real RAM
- pseudopolynomial number of steps
  - ↳ poly. in n & r = max. dist./min. distance

Approach:

- define energy function on configurations:

$$E(C) = \sum_{\text{edge } vw} \ \sum_{\substack{\text{vertex } u \\ \neq v \text{ or } w}} \frac{1}{d(u, vw)}$$



- any energy-decreasing motion avoids crossings: approaching 0 dist. shoots $E \to \infty$
- expansive motion decreases energy (in fact, every term)

⇒ energy-decreasing motions exist

⇒ downhill gradient of energy exists $-\nabla E$
  - computable in $O(n^2)$ time

- lower bound gradient, upper bound curvature

⇒ $O(n^{123} r^{41})$ step bound (!)

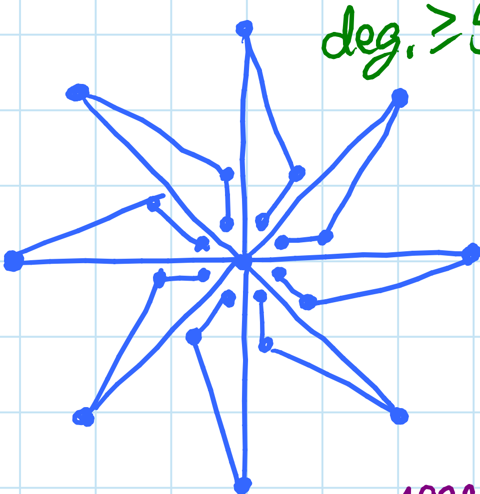
OPEN: improve step bound (likely not hard)

OPEN: $n^{O(1)}$ step bound possible? conjecture no

OPEN: is minimum-energy configuration unique? for equilateral polygons, it's a regular n-gon

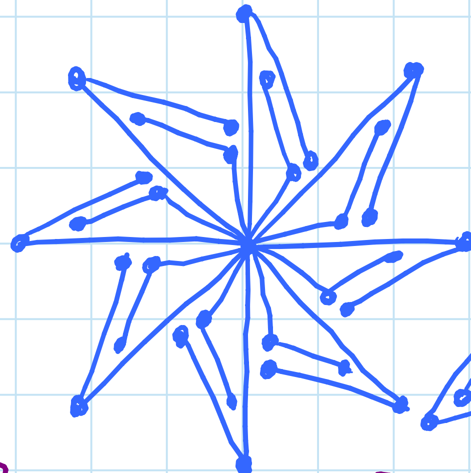
Locked 2D trees:

deg. ≥ 5



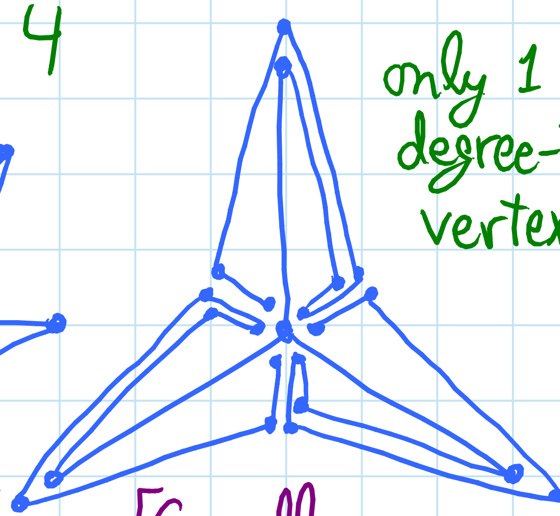
[Biedl et al. 1998/2002]

diameter 4



[Poon 2005]

only 1 degree-3 vertex



[Connelly, Demaine, Rote 2002]

$\boxed{\text{OPEN}}$: locked equilateral trees, 2D or 3D? [Poon; Demaine]
- no locked equilat. 2D trees of diameter ≤ 4 [Poon 2005]

$\boxed{\text{OPEN}}$: locked orthogonal trees in 2D? [Poon]
- no locked equilat. ortho. trees in 2D/3D [Poon 2006]

$\boxed{\text{OPEN}}$: characterize locked linkages
e.g. locked trees in 2D or chains in 3D
- polynomially solvable?

Related problem: can you fold config. A → config. B?
- PSPACE-complete for 2D trees & 3D chains [Alt, Knauer, Rote, Whitesides 2004]
- but their reductions use locked linkages as gadgets — so all locked

Infinitesimally locked linkages [Connelly, Demaine, Rote 2002]

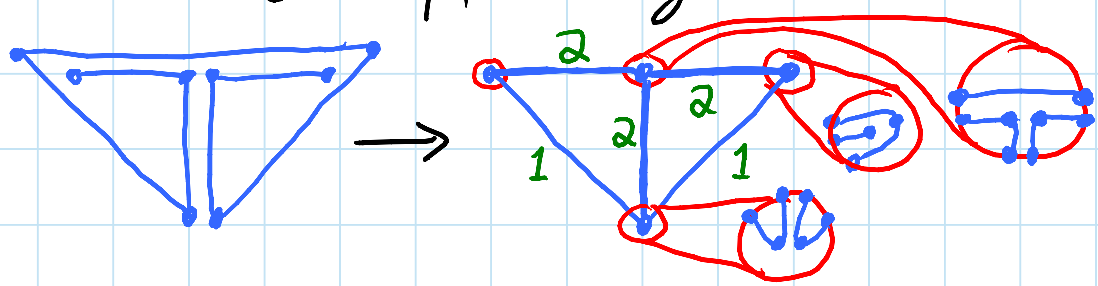
Intuition: in many locked examples (particularly 2D), as gaps get smaller, so do valid motions

Locked within ε = configuration from which it is impossible to get farther than ε in configuration space

Rigid = locked within $\emptyset$

— but trees are never rigid... right?

Self-touching configuration allows infinitesimal gaps: geometric overlap, distinguished combinatorially



— now can be rigid

Return to nontouching: rigidity ⇒ "strongly locked"

Strongly locked = sufficiently small perturbations are locked within ε, for any $\varepsilon > 0$

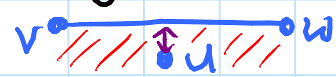
δ-perturbation = move vertices within δ-disks, preserving combinatorial sidedness

Every self-touching has a (non-self-touching) δ-perturbation, for all $\delta > 0$ [Ribó Mor, PhD 2006]

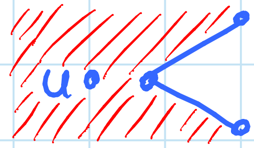
Infinitesimally locked linkages: (cont'd)

Infinitesimal rigidity:

- implies rigidity
- "zero-length strut" (linear inequality):
  u should remain right of vw



- sometimes nonconvex:

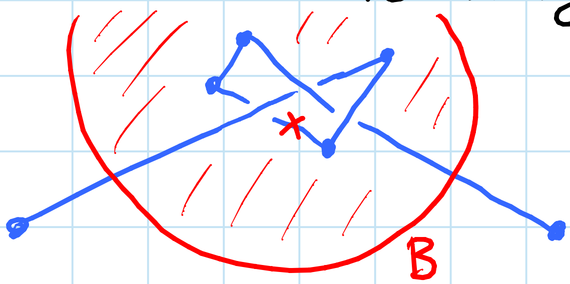


⇒ conservative polynomial test (drop constraints)
  or exponential test (split into 2 convex)

- analogs of equilibrium stress & duality
- even Maxwell-Cremona [Ribó Mor, PhD 2006]
- nice proofs by hand: positive stress on struts
  + underlying linkage rigid
  (⇒ inf. rigid
  ⇒ rigid
  ⇒ strongly locked)

PROJECT: implement locked linkage tester/designer tool

3D knitting needles: locked if each end bar is longer than Σ middle bars



[Cantarella & Johnston 1998]

Proof: draw ball B centered at midpoint of middle bars, diameter $= \Sigma$ middle bars $+ \varepsilon$

$\Rightarrow$ middle vertices remain inside B, end vertices remain outside B.

$\Rightarrow$ any motion could be augmented by an unknotted rope connecting two ends outside B.

$\Rightarrow$ straightening motion would untie trefoil knot. $\square$

$\boxed{\text{OPEN}}$: minimum possible edge length ratio for which locked 3D chain exists?

- best example is $1 : 3 + \varepsilon$ above

$\boxed{\text{OPEN}}$: any locked equilateral 3D chain? [Biedl et al.]

equilateral 3D chain self-weaving on line [E. Demaine]

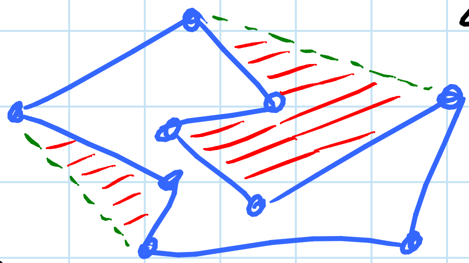
equilateral unknotted closed chains? [M. Demaine]

equilateral trees? [E. Demaine; Poon]

equilateral chain of equal-width cylinders? [O'Rourke]

Flips & friends:

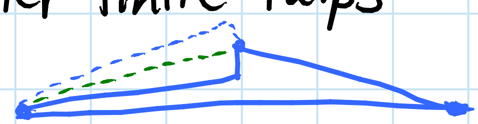
Pocket of 2D polygon = region outside polygon
& inside convex hull



Pocket lid = convex-hull edge

Flip = reflect pocket through its lid
= rotate 180° through 3D around the lid
- avoids self-intersection (line of support)
- increases area

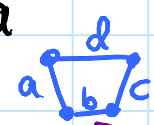
"Erdős-Nagy" Theorem: [posed by Erdős 1935]
[wrong proof by Nagy 1939]
[Demaine, Gassend, O'Rourke, Toussaint 2006]
any polygon always convexifies after finite flips
- but can be arbitrarily many:



[Joss & Shannon 1973]
- $\boxed{\text{OPEN}}$: bound # flips in n & r = max. dist./min. dist
  - pseudopolynomial? [Overmars 1998]

Deflation: inverse of flip (avoiding crossings)

- quadrilaterals with $a+c=b+d$ & $a \neq b \neq c \neq d \neq a$ always deflate infinitely

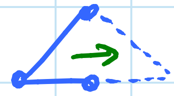


  [Fevens, Hernandez, Mesa, Morin, Soss, Toussaint 2001]
- that's all such quads [Ballinger, PhD 2003]
- no pentagon always deflates infinitely
  [Demaine, Demaine, Souvaine, Taslakian 2007]
- $\boxed{\text{OPEN}}$: any $n \geq 6$ gon with no flat vertices that always deflates to flat limit?
- $\boxed{\text{OPEN}}$: does any infinite deflation sequence induce a unique limit polygon?
- $\boxed{\text{OPEN}}$: characterize infinitely deflatible polygons
  - algorithm for testing a sequence?

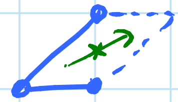
Flipturn: rotate pocket 180° in 2D around lid midpoint

- at most $n!$ configurations [Joss & Shannon 1973]
- always $O(n^2)$ flipturns [Aichholzer et al. 2002; Ahn et al. 2000 (diff. model)]
- sometimes $\Omega(n^2)$ flipturns [Biedl 2004]
- final polygon & location determined
- NP-hard to find longest flipturn sequence
- $\boxed{\text{OPEN}}$: finding shortest flipturn sequence?

[Aichholzer et al. 2002]

Pop: flip on 2 incident edges 

- can be forced to introduce crossings ⇒ allow
- OPEN : possible to convexify any polygon in finitely many pops? [Ballinger & Thurston 2001]

Popturns: flipturn on 2 incident edges 

- can convexify any polygon allowing self-intersection
- characterization of when possible without crossing

[Aloupis et al. 2007]