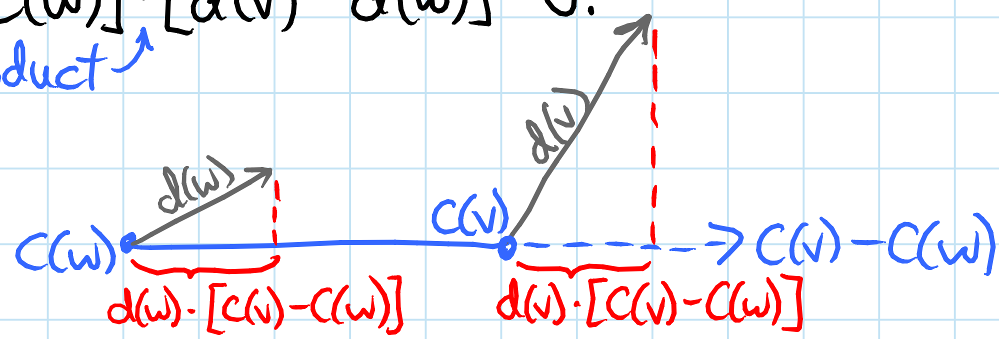
6.885	Lecture 4	Sept. 17, 2007

Infinitesimal rigidity: rigidity to the first order

Infinitesimal motion of a linkage configuration
= valid first derivative of a motion w.r.t. time, at time ∅
= velocity vector $d(v)$ for each vertex v
preserving edge lengths to the first order:

$$[C(v) - C(w)] \cdot [d(v) - d(w)] = 0.$$

$a \cdot b \longrightarrow$ dot product
$= a_x \cdot b_x + a_y \cdot b_y$



Rigidity matrix: "everything is linear, to the first order"
edge-length constraints form a linear system:

$$\text{row per edge}\left\{\underbrace{\begin{pmatrix} 0 \cdots 0 \,|\, C_x(v)-C_x(w) \,|\, C_y(v)-C_y(w) \,|\, 0 \cdots 0 \,|\, C_x(w)-C_x(v) \,|\, C_y(w)-C_y(v) \,|\, 0 \cdots 0 \end{pmatrix}}_{dn \text{ columns } (d \text{ dim., } n \text{ vertices})} \cdot \begin{pmatrix} d_x(v_1) \\ d_y(v_1) \\ \vdots \\ d_x(v_n) \\ d_y(v_n) \end{pmatrix} = \emptyset\right.$$

RIGIDITY MATRIX R

Infinitesimal motions $= d$ for which $R \cdot d = \emptyset$
$=$ kernel R $=$ nullspace R
$\leftrightarrow$ linear subspace of some dimension: nullity R

Infinitesimally rigid if nullity $R = \frac{d(d+1)}{2}$ ← rigid motions

Rank-Nullity Theorem: rank R + nullity R = # cols. $= d \cdot n$

$\Rightarrow$ inf. rigid $\Leftrightarrow$ rank $R = d \cdot n - \frac{d(d+1)}{2}$ "full rank"

$\Rightarrow$ can test inf. rigidity in polynomial time using e.g. Gaussian elimination

Generic point set (more explicit definition than L3)

= all minors of rigidity matrix of complete graph

induced square submatrix on subset of rows & cols.

with nonzero determinant for some point set (i.e. not identically zero, algebraically) are nonzero for this point set

Generic results:

- almost every configuration is generic
- at generic configurations,
  rigidity = infinitesimal rigidity = generic rigidity [L3]

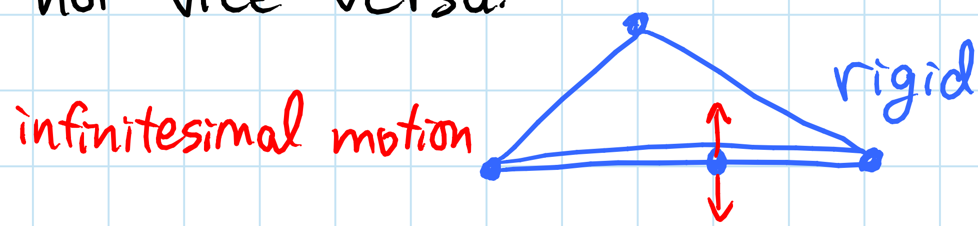
$\Rightarrow$ randomized polynomial-time generic rigidity test:
test infinitesimal rigidity of random realization

- if any realization is infinitesimally rigid then graph is generically rigid
  (else generically flexible with probability 1)

Taking derivatives: flexible $\Rightarrow$ infinitesimally flexible

i.e. infinitesimally rigid $\Rightarrow$ rigid

- but not vice versa:

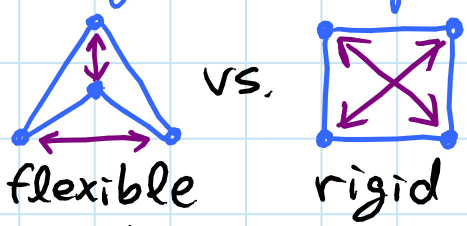


Tensegrity = tens(ional int)egrity [Snelson 1968; R. Buckminster Fuller]

$\boxed{\text{PROJECT}}$: build tensegrity sculpture

= linkage but where each edge is either:
- bar (as before): fixed length
- cable: length can only decrease (string/elastic/spider web)
- strut: length can only increase

- configuration space becomes $\underbrace{\text{semi-algebraic}}_{\text{polynomial inequalities}}$
- motion, rigidity as before
- but not generic rigidity:



- infinitesimal motion (& rigidity):

$$[C(v)-C(w)]\cdot[d(v)-d(w)] \begin{cases} = 0 & \text{for bars } vw \\ \le 0 & \text{for cables } vw \\ \ge 0 & \text{for struts } vw \end{cases}$$

$\Rightarrow$ inf. motion space is a polyhedral cone

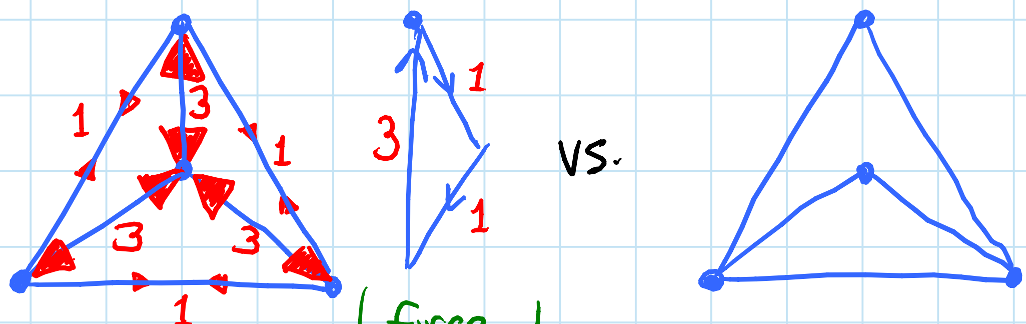
$\Rightarrow$ inf. rigidity testable in polynomial time via linear programming

<u>Equilibrium stress</u> = real number for each edge $s: E \to \mathbb{R}$
such that $s(e) \geq 0$ for cables e (push back in resistence)
$s(e) \leq 0$ for struts e

EQUILIBRIUM → $\displaystyle\sum_{\text{edge } vw} s(vw) \cdot [C(v) - C(w)] = 0$
for all vertices v.

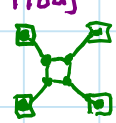
- view $s(vw)$ as a <u>scale factor</u> on force along edge vw felt equally by v & w.
  - $s(vw) > 0 \Rightarrow$ push on v & w (resist compression)
  - $s(vw) < 0 \Rightarrow$ pull on v & w (resist expansion)
  - $s(vw) = 0 \Rightarrow$ no force
- <u>trivial</u> equilibrium stress: $s(e) = 0$ for all e

<u>Example:</u>



equilibrium stress | force polygon | no nontrivial equi. stress

<u>Duality in tensegrities:</u> [Roth & Whiteley 1981]
- some equilibrium stress is nonzero on strut/cable e
  $\Leftrightarrow$ every infinitesimal motion holds e's length fixed
- tensegrity is infinitesimally rigid
  $\Leftrightarrow$ every strut/cable is nonzero in some equilibrium stress
  & <u>corresponding linkage</u> is rigid   } not necessary for "spiderweb" [Connelly 1982]
  ↳ replace cables & struts with bars
- proofs based on linear-programming duality

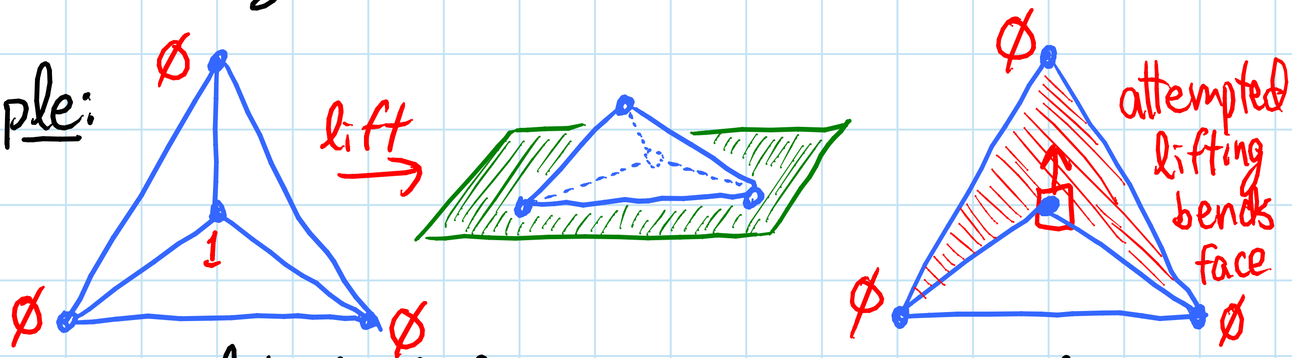


Polyhedral lifting of a noncrossing configuration

= z coordinate for each vertex $z: V \to \mathbb{R}$
such that each face remains planar

- assume outside face at $z=0$ by rigid motion
- trivial lifting: $z(v)=\emptyset$ for all v

Example:

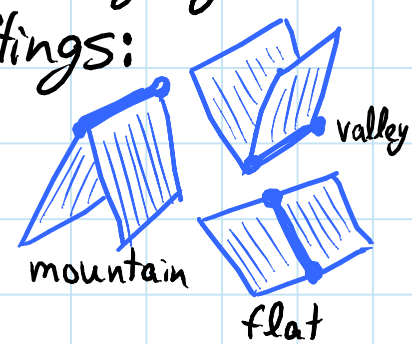


polyhedral lifting   no nontrivial poly. lifting

Maxwell–Cremona Theorem: [Maxwell 1864; Cremona 1872]

one-to-one correspondence in noncrossing tensegrity between
equilibrium stresses & polyhedral liftings:

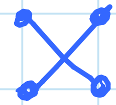
- negative stress ⇔ valley edge
- positive stress ⇔ mountain edge
- zero stress ⇔ flat edge



PROJECT: implement program to illustrate
stress ⇔ lifting correspondence
and/or stress ⇔ inf. motion correspondence

PROJECT: virtual tensegrity building toy
- illustrate infinitesimal flexibility if any

Noncrossing linkages:

- configuration cannot have crossing edges 
- config. space smaller; still semi-algebraic

Locked linkage if config. space is disconnected
i.e. no motion between some two configurations

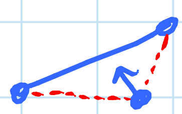
- summary: [L1]

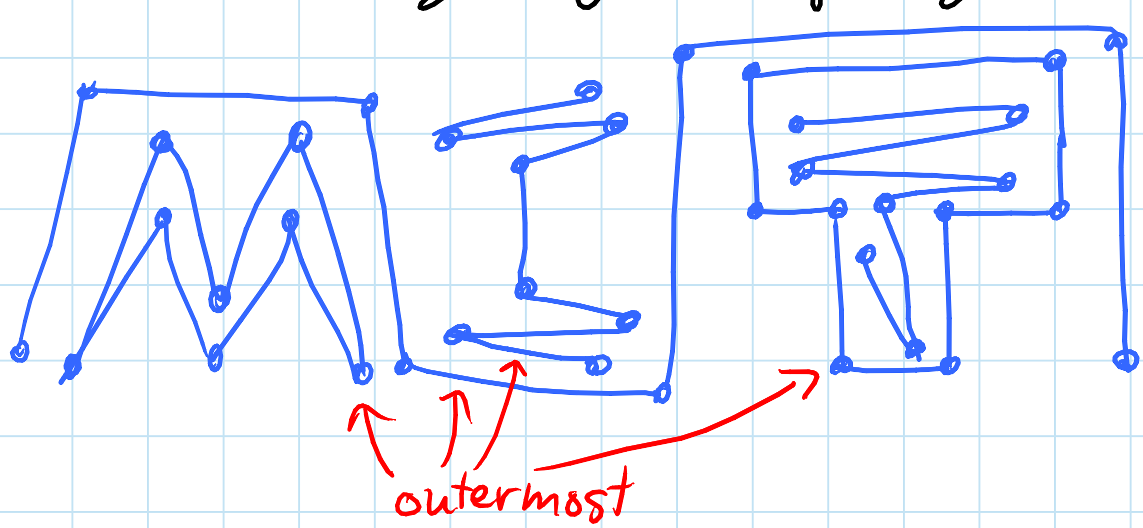
	chains	trees
2D	never locked	can lock
3D	can lock	can lock
4D+	never locked	never locked

Carpenter's Rule Theorem: [Connelly, Demaine, Rote 2000/2003]

any linkage configuration of maximum degree 2
has a motion that

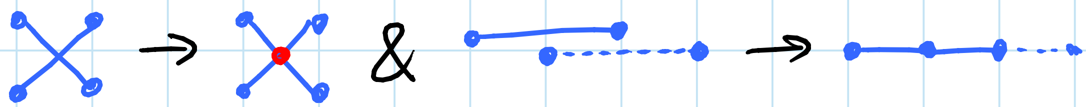
- straightens/convexifies all outermost open/closed chains
- is expansive: distance between any two vertices only increases

$\Rightarrow$ is noncrossing, by Δ inequality 

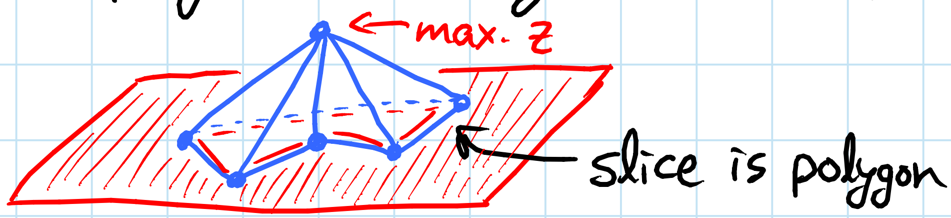


Proof sketch of Carpenter's Rule Theorem:

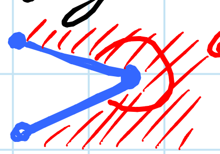
- build tensegrity from linkage (edges → bars)
  + all possible struts (except where bar exists)
- linkage has expansive infinitesimal motion
  ⇔ tensegrity is infinitesimally flexible
  ⇐ every equilibrium stress is zero (on struts)
  ⇔ every polyhedral lifting is flat (on struts)
  - detail: need to show stresses are equivalent in tensegrity vs. planarized tensegrity



  - here is where nested components get discarded
- slice hypothetical polyhedral lifting near max. z:
  - peak case:



    - convex vertices ⇔ mountain edges
    - reflex vertices ⇔ valley edges
    - every polygon has ≥3 convex vertices
    ⇒ ≥3 incident mountains (positive stress)
    ⇒ ≥3 incident bars (no cables)
    - but max. degree 2
  - general case:



flat: need convex vertex to reach around reflex angle

    ⇒ flat except inside convex polygons
- integrate ordinary differential equation → expansive motion