6.841 Advanced Complexity Theory March 9, 2009

Lecture 10

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Meeting to talk about final projects on Wednesday, 11 March 2009, from 5pm to 7pm. Location: TBA. Includes food.

1 Overview of today's lecture

- Randomized computation.
- Complexity classes: RP, coRP, BPP, ZPP.
- Basic properties of these complexity classes.

So far, we know that P is a computationally feasible class. We could try and expand this notion, and then study where the expanded notions lie in relation with P, NP, etc.

2 Examples of problems which have randomized algorithms

1. **Problem:** Find an n-bit prime.
   Input: $N \in \mathbb{N}$, $N > 3$ such that $2^{n-1} < N \leq 2^n$.
   Output: A prime p, such that $N \leq p < 2N$.

   A polynomial-time algorithm for this problem is as follows. This algorithm is randomized. No deterministic algorithm is known.

   ```
   loop {n times}
     Pick k randomly and uniformly between N (inclusive) and 2N (exclusive).
     if k is prime then
       return k
     else
       continue loop.
   return a random value between N (inclusive) and 2N (exclusive).
   ```

   A sketch of the proof of correctness of this algorithm is as follows.

   Sketch of Proof First we observe that we can always find such a prime. This is the following lemma, which we state without proof.

   Lemma 2.1 (Bertrand's Postulate) *If n is a natural number greater than 3, then there exists a prime number p such that $n \leq p < 2n$.*

   Apart from Lemma 2.1 the algorithm depends on the Prime Number Theorem, which we state without proof.

   Theorem 2.2 (Prime Number Theorem) *For any real number x, let $\pi(x)$ be the number of primes less than or equal to x. Then,*

   $$\lim_{x \to \infty} \frac{\pi(x)}{x / \ln x} = 1.$$

In this context, the Prime Number Theorem implies that the number of primes between N and $2N$ is about $\frac{2N}{n+1} - \frac{N}{n}$, which is approximately $\frac{N}{n}$. So the probability of k being prime is approximately $\frac{1}{n}$. Since the algorithm is repeated n times, the probability of it not returning a prime is approximately

$$\left(\frac{n-1}{n}\right)^n = \left(1 - \frac{1}{n}\right)^n \leq \frac{1}{e}.$$

We will see later that this error probability is small enough for our purposes. ■

2. **Problem:** Square-root modulo primes.
   Input: An n-bit long prime p, an integer a such that $0 \leq a \leq p$.
   Output: An integer α such that $\alpha^2 = a \pmod{p}$.

   Berlekamp, and later Adleman, Manders and Miller, gave randomized polynomial-time algorithms to solve this problem. A deterministic polynomial-time algorithm is not known.

   A randomized polynomial-time algorithm to solve this problem is as follows. First, β is chosen randomly and uniformly from $[p-1]$. If we can solve the equation $\gamma^2 = \beta^2\alpha \pmod{p}$ for γ, then $\alpha = \beta/\gamma$. For this, θ is picked randomly and uniformly from $[p-1]$, and the following equation can be solved, $(x-\theta)^2 = \beta^2\alpha \pmod{p}$. To do this, we use (without proof) the fact that $\gcd(x^2 - 2x\theta + \theta^2 - \beta^2\alpha, x^{\frac{p-1}{2}} - 1)$ is linear in x with probability $1/2$.

   If we find this gcd q and if it is linear in x, then it will be either $x - \theta - \beta\alpha$ or $x - \theta - \beta\alpha$, so we can just return $(x - \theta - q)/\beta$.

3. **Problem:** Given k $n \times n$ square matrices of integers $M_1, M_2, \ldots, M_k$, do there exist integers $r_1, r_2, \ldots, r_k$ such that $\det(\sum_{i=0}^{k} r_i M_i) \neq 0$?

   A randomized algorithm for this problem is as follows. Pick $r_1, r_2, \ldots, r_k$ randomly and uniformly from $\{1, 2, \ldots, 3n\}$ and check if $\det(\sum_{i=0}^{k} r_i M_i) \neq 0$. If so, output 'yes'; otherwise output 'no'.

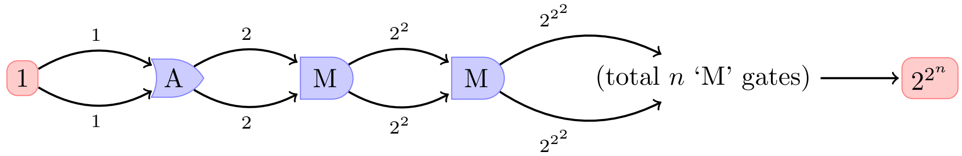
   The proof of the correctness of this algorithm is discussed in Section 3.

4. **Problem:** Equivalence of circuits.
   Input: Circuits C_1, C_2 over integer inputs $x_1, x_2, \ldots, x_n$ with addition and multiplication gates and the constants $\{-1, 0, 1\}$.
   Output: Is C_1 equivalent to C_2? (Is the function computed by C_1 the same as the function computed by C_2?)

   A randomized algorithm for this problem is as follows. (Here, we assume that the size of the circuit is a polynomial in the number of inputs, to make estimations about the input size of our problem simpler.)

   Pick a prime of size about $2^{O(n)}$ and call it p.
   Pick $x_1, x_2, \ldots, x_n$ randomly and uniformly in $\mathbb{Z}_p$.
   {In the following if-statement, the output of each gate is computed in $\mathbb{Z}_p$.}
   if $C_1(x_1, x_2, \ldots, x_n) = C_2(x_1, x_2, \ldots, x_n)$ **then**
       return 'yes'
   else
       return 'no'

   Observe that we can have a polynomial-sized circuit that computes 2^{2^n}, as follows. (Here, 'A' denotes addition and 'M' denotes multiplication).



The number 2^{2^n} is too big for polynomial-time simulations, and it is clear that we can actually get a number of this size from a circuit of polynomial size. So we reduce modulo p, so as to restrict all the possible numbers in our computations to have at most $O(n)$ bits. This ensures that the algorithm does not exceed polynomial time.

3 Some proofs

To prove the algorithms for finding square-root modulo primes and for checking the equivalence of two circuits, we will use the following lemma. Recall that we have already used once in a previous lecture.

Lemma 3.1 (Schwarz-Zippel Lemma) *Let $p(x_1, \ldots, x_n)$ be a not identically zero polynomial of total degree d over any (possibly infinite) field $\mathbb{F}$. If $\alpha_1, \ldots, \alpha_n$ are chosen uniformly at random from any finite set $S \subset \mathbb{F}$, then*

$$\Pr\left[p(\alpha_1, \ldots, \alpha_n) = 0\right] \leq \frac{d}{|S|}.$$

To prove Item 2 (Square-root modulo primes), we have to calculate the probability that the algorithm makes an error. Note that $p(x_1, \ldots, x_k) = \det(\sum_{i=0}^{k} x_i M_i)$ is a polynomial of degree $d = n$ in the variables x_i. Suppose that the polynomial is not identically zero, otherwise the algorithm can never err.

If with the randomly chosen r_i, $\det(\sum_{i=0}^{k} r_i M_i)$ turns out to be non-zero, then there certainly exists an assignment to the x_is such that $p(x_1, \ldots, x_i)$ is non-zero. On the other hand, if with the randomly chosen r_i the quantity $p(r_1, \ldots, r_i)$ is zero, then there is some probability of error. This can be calculated by using Lemma 3.1. We have chosen the set S to be $\{1, 2, \ldots, 3n\}$, and the r_i have been chosen randomly and uniformly from S. Therefore,

$$\Pr\left[p(r_1, \ldots, r_k) = 0\right] \leq \frac{d}{|S|} = \frac{n}{3n} = \frac{1}{3}.$$

We will only sketch the proof of Item 4 (Circuit equivalence). For this we need to estimate the error probability. If for some choice of $x_1, x_2, \ldots, x_n$, we get that $C_1(x_1, \ldots, x_n) \neq C_2(x_1, \ldots, x_n)$ modulo p, then the circuits are certainly not equivalent. On the other hand, if $C_1(x_1, \ldots, x_n) = C_2(x_1, \ldots, x_n)$ modulo p, then there is some probability of error. We can also represent $C_1(x_1, \ldots, x_n)$ and $C_2(x_1, \ldots, x_n)$ as polynomials in $x_1, \ldots, x_n$. The following facts lead to the proof that the probability of error is sufficiently small.

- The degrees of the polynomials corresponding to the circuits may be quite large, but Lemma 3.1 still works because $|S| = |\mathbb{Z}_p| \geq 2^{\Omega(n)}$.
- The numbers appearing during the computation are no more than n bits long after reduction modulo p. If the original probability of error is ϵ, then we can repeat this algorithm poly(n) times to decrease this to $\epsilon^{\text{poly}(n)}$, by using the following well-known result.

  Theorem 3.2 (Chinese Remainder Theorem) *Let M, N be integers such that for each prime p_i from k distinct primes $p_1, p_2, \ldots, p_k$, $M \equiv N \,(\mathrm{mod}\ p_i)$. Then $M \equiv N \,(\mathrm{mod}\ p_1 p_2 \cdots p_k)$.*

4 Complexity classes

4.1 Types of randomized algorithms

To start off the discussion of complexity classes, we first consider the kinds of errors that may occur in a randomized algorithm. For the purposes of the discussion below, we fix $\epsilon = \frac{1}{3}$. We will see later that this particular choice of ϵ is not special. Let L be a language, and suppose that our algorithm is deciding membership of x in L. Then we can have the following types of errors.

1. Two-sided error:
   $x \in L \Rightarrow$ probability of error is at most ϵ, and
   $x \notin L \Rightarrow$ probability of error is at most ϵ.

   The class of polynomial-time algorithms that behave in this manner is called BPP, which stands for Bounded-error Probabilistic Polynomial-time.

2. One-sided error. There are two types of one-sided error:

   (a) $x \notin L \Rightarrow$ no errors, and
   $x \in L \Rightarrow$ probability of error is at most ϵ.

   The class of polynomial-time algorithms that behave in this manner is called RP, which stands for Randomized Polynomial-time.

   (b) $x \in L \Rightarrow$ no errors, and
   $x \notin L \Rightarrow$ probability of error is at most ϵ.

   The class of polynomial-time algorithms that behave in this manner is called coRP, which stands for co-Randomized Polynomial-time.

3. Zero-sided error:
   $x \in L \Rightarrow$ no error,
   $x \notin L \Rightarrow$ no error, but
   may not halt on some inputs.

   Alternatively, we can say that the running time of the algorithm is a random variable with polynomial expectation. Or, we can say that the algorithm is permitted to return one of three values, 1 if it accepts, 0 if it rejects, and ? if it does not know (within some fixed time).

   The class of polynomial-time algorithms that behave in this manner is called ZPP, which stands for Zero-error Probabilistic Polynomial-time.

4.2 Models of randomized computation

A natural model for randomized computation is a Turing Machine M which has a special 'coin-tossing' state, the '$' state.

However, the preferred model for randomized computation is that of *Two-input Turing Machines.* In this case, x is the real input and y is an auxiliary input. The input x represents the actual input data, and the input y captures the randomness used for a particular instance of a randomized computation. A Two-input Turing Machine M takes in (x, y) and runs deterministically on (x, y). (For the cases of RP, coRP, ZPP and BPP, M must run in polynomial-time of the input, and therefore y must be a polynomial in the size of x).

4.3 New definitions for complexity classes

Using the language of two-input Turing Machines, we can redefine some of the complexity classes that we already know. Again, $\epsilon = \frac{1}{3}$. For each of these complexity classes, the language L is in the class if there is a two-input Turing Machine M with the second input always a polynomial in the size of the first, such that certain results (defined in the following list) are true. In the following experiments, y is always chosen uniformly from all the available possibilities.

1. BPP

   (a) If $x \in L$ then $\Pr_y [M(x, y) \text{ accepts}] \geq 1 - \epsilon$.

   (b) If $x \notin L$ then $\Pr_y [M(x, y) \text{ accepts}] \leq \epsilon$.

2. NP

(a) If $x \in L$ then $\Pr_y[M(x,y) \text{ accepts}] > 0$.

(b) If $x \notin L$ then $\Pr_y[M(x,y) \text{ accepts}] = 0$.

3. RP

   (a) If $x \in L$ then $\Pr_y[M(x,y) \text{ accepts}] \geq 1 - \epsilon$.

   (b) If $x \notin L$ then $\Pr_y[M(x,y) \text{ accepts}] = 0$.

4. coRP

   (a) If $x \in L$ then $\Pr_y[M(x,y) \text{ accepts}] = 1$.

   (b) If $x \notin L$ then $\Pr_y[M(x,y) \text{ accepts}] \leq \epsilon$.

5. ZPP
   ZPP cannot be naturally defined with this notation, so we can give the following definition.

$$\mathrm{ZPP} = \mathrm{RP} \cap \mathrm{coRP}.$$

5 Choice of error parameter

What is the ideal choice for the maximum permissible error? Is it on the order of $1/3$, $1/n^3$, $1/2^n$, or on the order of $1 - 1/n^5$, $1 - 1/2^n$? Let us only look at the class RP for now. This can be formalized by looking at the following result.

Lemma 5.1 (Amplification Lemma) *Suppose an algorithm M errs with probability $e(n)$, so that if $x \in L$ then $Pr_y[M(x,y)$ accepts$] \geq 1 - e(n)$ and if $x \notin L$ then $Pr_y[M(x,y)$ accepts$] = 0$ (when y is chosen uniformly). Repeat M $k(n)$ times, for some polynomial k. The new algorithm makes errors with the following probabilities.*

$$x \in L \Rightarrow Pr_y[M(x,y) \text{ accepts}] \geq 1 - (e(n))^{k(n)},$$
$$x \notin L \Rightarrow Pr_y[M(x,y) \text{ accepts}] = 0.$$

RP with an error probability of $e(n)$ may be written as $\mathrm{RP}_{e(n)}$.

If we start with a constant error probability, we can make it as small as $1/2^{n^c}$ for any c in polynomially many iterations of the algorithm. This probability is small enough.

If we start with an error probability $e(n) = 1 - 1/n^5$, then

$$(e(n))^{k(n)} = \left(1 - \frac{1}{n^5}\right)^{n^5 l(n)} \leq \left(\frac{1}{e}\right)^{l(n)},$$

which is small enough if $k(n)$ is a sufficiently large degree polynomial.

But if we start with an error probability $e(n) = 1 - 1/2^n$, then we cannot make the error probability small enough after polynomially many iterations of the algorithm. In this case, $\mathrm{RP}_{e(n)} = \mathrm{NP}$.

But $\mathrm{RP}_{1-1/\mathrm{poly}(n)} = \mathrm{RP}_{1-1/2^{\mathrm{poly}(n)}}$. So the class RP is robust with respect to large changes in the maximum permissible error probability.