Problem Set 3

Problems

1. **Promise-RP and Promise-BPP**

   (a) Define the promise classes Promise-RP (prRP) and Promise-BPP (prBPP) so that they have complete (promise) problems, and so that they include RP and BPP (respectively).

   (b) Define $\text{prRP}^{\text{prRP}}$ and $\text{prBPP}^{\text{prBPP}}$.

   (c) Prove that $\text{prBPP}^{\text{prBPP}} \subset \text{prBPP}$.

   (d) Prove that $\text{prRP}^{\text{prRP}} = \text{prBPP}$. *Hint: Review the proof of $\text{BPP} \subset \Sigma^2$.*

   (e) Conclude that if $P = \text{prRP}$, then $P = \text{prBPP}$.

2. **The hierarchy does not have fixed-polynomial size circuits**

   Prove that for any fixed $k$, $\text{TIME}(n^k)/n^k$ (or roughly equivalently $\text{SIZE}(n^k)$), the class of functions computed by circuits of size $n^k$ does not contain the Polynomial Hierarchy. (*More credit for showing that lower levels of the hierarchy are not contained in $\text{TIME}(n^k)/n^k$. [Corrected 3/17/09].*)

3. **PH vs. PSPACE**

   Prove that there is an oracle $A$ such that $\text{PH}^A \neq \text{PSPACE}^A$. *Hint: Parity is not in $\text{AC}^0$.*

4. **Sparse NP-complete languages**

   A language $L$ is said to be sparse if there exists a $k$ such that for all sufficiently large $n$, $|L \cap \{0,1\}^n| < n^k$. Prove that if a sparse language is NP-complete, then the Polynomial Hierarchy collapses. (*More credit for collapsing it to lower levels.*)

5. **AM[2] = AM**

   Write out the explicit definition of the complexity classes $\text{BP} \cdot \exists \cdot \text{P}$ and $\text{BP} \cdot \exists \cdot \text{BP} \cdot \exists \cdot \text{P}$. Prove that these two classes are equal.