LECTURE 19

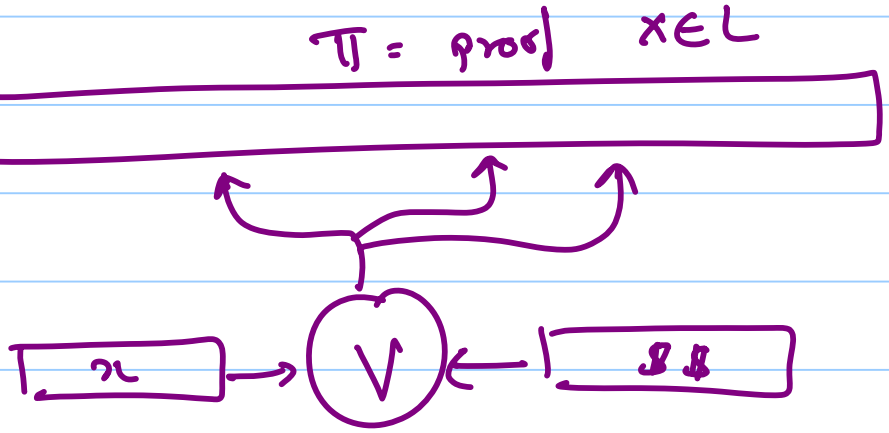
Note Title

TODAY: PCP's Contd.

- An Exponentially long PCP for Satisfiability.

Review of last lecture: 2 views of PCP

(1)



$x \in L \Rightarrow \exists \Pi \; Pr\left[V^{\Pi}(x,R)\right] \geq c(n)$

$x \notin L \Rightarrow \forall \Pi \; Pr\left[V^{\Pi}(x,R)\right] \leq s(n)$

② $L \leq$ Generalized Graph k-coloring

$x \in L \Rightarrow G_x$ k-colorable

$x \notin L \Rightarrow$ Every k-coloring violates ϵ-fraction of edges.

Today's result: Exponentially long proof verifiable w. $O(1)$ queries.

- Non-trivial in View 1
- Trivial in View 2

Key Ideas: - Arithmetization of SAT

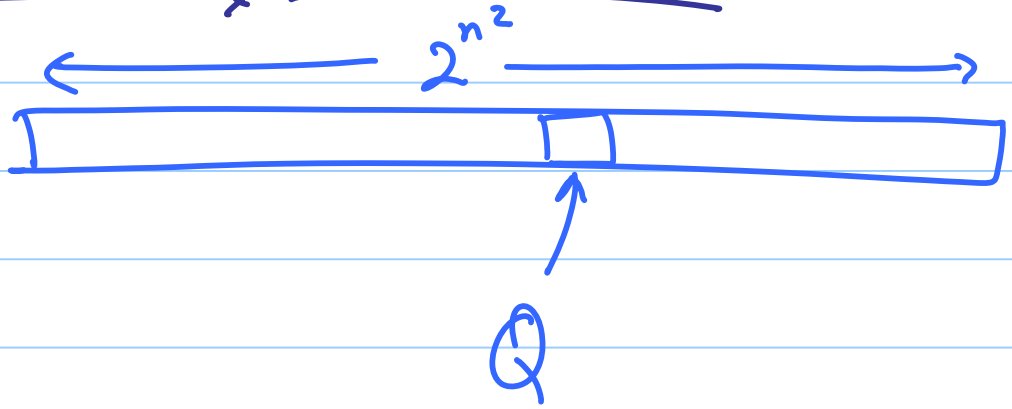
- Nice Properties of Linear functions; (low-degree polynomials).

L = Quadratic SAT

Input = $P_1 \dots P_m$ m poly of deg 2 in n variables.

Goal: $\exists (a_1 \dots a_n) = \bar{a}$ s.t. $P_1(\bar{a}) = P_2(\bar{a}) \dots P_m(\bar{a}) = 0$.

PCP: $\Pi =$



$\Pi[Q] = Q(a)$ $\forall$ quadratic functions Q

(so $Q(x_1 \dots x_n) = \sum_{\substack{1 \le i \le n \\ 1 \le j \le n}} q_{ij} x_i x_j + q_0$)

$Q \equiv (\{q_{ij}\}, q_0)$

How to check this proof?

Problem: Π may not equal $\{Q[a]\}_Q$ for any a.

Solution:

Test { Will ensure $\exists$ a s.t. $\Pi[Q] = Q(a)$ for most a. }

~~x~~

New Problem: How to test that for this a

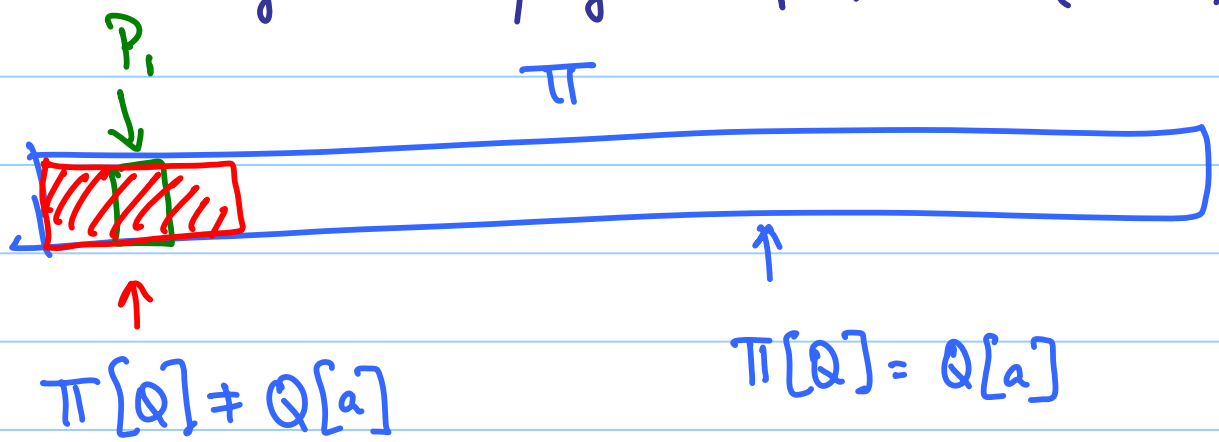
$P_1(a) = P_2(a) = \cdots P_m(a) = 0$?

New Solution

Will give prob. alg. to check this if Π satisfies Test.

Test (eol) $\Longrightarrow$ Satisfaction Test.

- Suppose only one poly P_1. $(m=1)$.



- Can't just read $\pi[P_1]$! Maybe wrong.
- But can compute $\tilde{\pi}(P_1)$

$$= \pi[P_1 + Q] - \pi[Q]$$

$$\| \qquad\qquad\qquad\qquad \uparrow \text{whp.}$$

$$(P_1+Q)(a) - Q(a) = P_1(a)$$

Formally: if $\Pr_Q[\pi[Q] \neq Q[a]] \leq \delta$

then $\Pr[\tilde{\pi}(P_1) \neq P_1(a)] \leq 2\delta$

- Many $P_1 \dots P_m$?

  - Can't repeat above for every m!
  - Any way to do $\bigwedge_{j=1}^{m} P_j$ efficiently?

    (algebraically, low-degree ?)

Solution: Pick $\alpha_1 \dots \alpha_m \in_U \{0,1\}$

& check $P_{\bar{\alpha}} = \sum \alpha_j P_j$

if $P_1 \dots P_m = 0$ then so is $P_{\bar{\alpha}}$

if $\exists j \; P_j \neq 0$ then $\Pr_{\bar{\alpha}}\left[P_{\bar{\alpha}}(a) \neq 0\right] \geq \frac{1}{2}$.

Back to "Test"! How to do it?

Goal: Given $\Pi = \{\Pi[Q]\}_Q$

Test: $\exists$ a s.t.

$$\Pr_Q\left[\Pi(Q) \neq Q(a)\right] \leq \delta .$$

Idea: Look for structural properties of such a Π & test for them.

Example: $\forall\ Q_1, Q_2$

$$(Q_1 + Q_2)(a) = Q_1(a) + Q_2(a)$$

Can test this

$$\Pi[Q_1 + Q_2] \stackrel{?}{=} \Pi[Q_1] + \Pi[Q_2] ?$$

- $\forall\ Q_1, Q_2$? cleary terrible !
- $O(n^2)$ times ? $O(n)$ times ? $O(1)$ times?

(for random Q_1, Q_2).

Remarkable Theorem [BLR]:

$$\Pr_{Q_1, Q_2}\left[\pi[Q_1] + \pi[Q_2] = \pi[Q_1+Q_2]\right] = 1-\epsilon > \frac{7}{9}$$

$\Rightarrow \exists\ \tilde{\pi}$ s.t. $\forall Q_1 Q_2$

$$\tilde{\pi}[Q_1] + \tilde{\pi}[Q_2] = \tilde{\pi}[Q_1+Q_2]$$

& $\Pr_{Q}\left[\pi[Q] \neq \tilde{\pi}[Q]\right] \leq 2\epsilon$.

Won't Prove Theorem!

What does $\tilde{\pi}$ look like?

$\Rightarrow \exists\ y_0, y_{ij}$ s.t.

$\forall\ Q = (\{q_{ij}\}, q_0)$; $\tilde{\pi}[Q] = \sum q_{ij} y_{ij} + q_0 y_0$

Are we done ?

Still need to test

① $y_{ij} = x_i \cdot x_j \qquad \forall i, j$

② $y_0 = 1$

② Easy:

Verify $\bar{1}(a) = 1 \qquad (\forall a)$

$\bar{1} = \{\{0\}, 1\}$

$\tilde{\pi}[\bar{1}] = 1 \approx \pi[Q + \bar{1}] - \pi[Q] = 1$?

III. Satisfiability of $P_1 \dots P_m$

- $\bar{\alpha} = (\alpha_1 \dots \alpha_m)$ $\longleftarrow$ random ; $Q \leftarrow$ random
- $P_{\bar{\alpha}} = \sum \alpha_j P_j$

$$\pi[Q + P_{\bar{\alpha}}] - \pi[Q] = 0 \quad ?$$

Analysis:

1. Makes $|Q| = O(1)$ queries;
2. $(\exists a \text{ s.t. } P_j(a) = 0 \ \forall j)$
   $\Rightarrow \exists \pi, \pi_{lin}$ s.t. $Pr[\text{Verifier}] = 1$
3. $Pr[\text{Verifier accepts}] \geq .99$
   $\Rightarrow \exists a$ s.t. $P_1(a) = \dots = P_m(a) = 0$

- Continuing optimism ....

Pick random W, V

$$\begin{bmatrix} w^T & \end{bmatrix} \begin{bmatrix} Y \end{bmatrix} \begin{bmatrix} v \end{bmatrix} = \begin{bmatrix} w^T & \end{bmatrix} \begin{bmatrix} X \end{bmatrix} \begin{bmatrix} X^T \end{bmatrix} \begin{bmatrix} v \end{bmatrix}$$

Claim: if $Y \neq XX^T$ then

$$\Pr_{v,w} \left[w^T Y v \neq w^T X X^T v \right] \geq \frac{1}{4}$$

Proof: Exercise

So how to test ...

$$\underbrace{\begin{bmatrix} w^T & \end{bmatrix} \begin{bmatrix} Y \end{bmatrix} \begin{bmatrix} v \end{bmatrix}}_{\tilde{\Pi}[Q_{v,v}]} = \begin{bmatrix} w^T & \end{bmatrix} \begin{bmatrix} X \end{bmatrix} \begin{bmatrix} X^T \end{bmatrix} \begin{bmatrix} v \end{bmatrix}$$

$\tilde{\Pi}[Q_{v,v}]$, $Q_{v,v} = \{q_{ij}\}$, $q_{ij} = v_i w_j$.

- $w^T x = L_w(a_1 \dots a_n) := \sum w_i a_i$.
- Ask prover to write $L(\bar{a})$ $\forall$ linear function L.

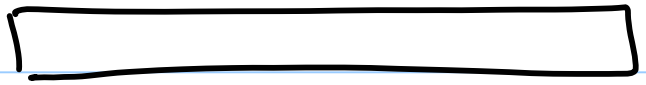
  & test by testing

$$\Pi_{lin}[L_1] + \Pi_{lin}[L_2] = \Pi_{lin}[L_1 + L_2].$$

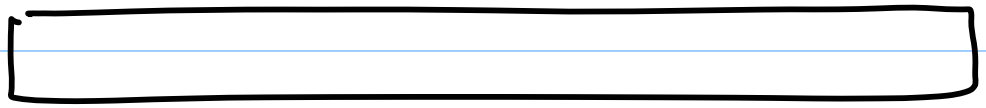
Summary

To prove $\exists\, a$ s.t. $P_1(a) = P_2(a) = \cdots P_m(a) = 0$.

PCP =



π_{lin}



π

(Supposedly $\pi_{lin}[L] = L(a)$ ← linear

$\pi[Q] = Q(a)$ ← quad. Q)

VERIFIER:

I. (Linearity of Π_{lin})

Pick L_1, L_2 at random & check

$$\Pi_{lin}[L_1] + \Pi_{lin}[L_2] = \Pi_{lin}[L_1 + L_2]$$

II. (Quadraticity of Π)

IIa: $Q_1, Q_2 \leftarrow$ random

$$\Pi[Q_1] + \Pi[Q_2] = \Pi[Q_1 + Q_2]$$

IIb: $L_1, L_2 \leftarrow$ random ; $Q \leftarrow$ random

$$\Pi[Q + L_1 \cdot L_2] - \Pi[Q] = \Pi_{lin}[L_1] \cdot \Pi_{lin}[L_2]$$

IIc: $Q \leftarrow$ random

$$\Pi[Q + \bar{1}] - \Pi[Q] = 1$$

III. Satisfiability of $P_1 \dots P_m$

- $\bar{\alpha} = (\alpha_1 \dots \alpha_m)$ $\longleftarrow$ random ; $Q \leftarrow$ random
- $P_{\bar{\alpha}} = \sum \alpha_j P_j$

$$\pi[Q + P_{\bar{\alpha}}] - \pi[Q] = 0 \quad ?$$

Analysis:

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Conclusions:

- Non-trivial PCPs exist.
- But no use in approximation?
- Turns out protocol has use, though we don't know how to use PCP directly.