LECTURE 08

Note Title

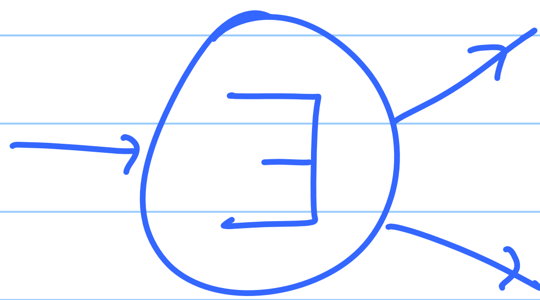
TODAY :
- ALTERNATION VS. TIME VS. SPACE
- FORTNOW'S THEOREM :

$$SAT \notin \bigcup_{c} TIME(n \log^{c} n)$$

$$\text{OR} \quad SAT \notin L$$

ALTERNATION :

ATM : Consists of a Turing Machine with 2 special states



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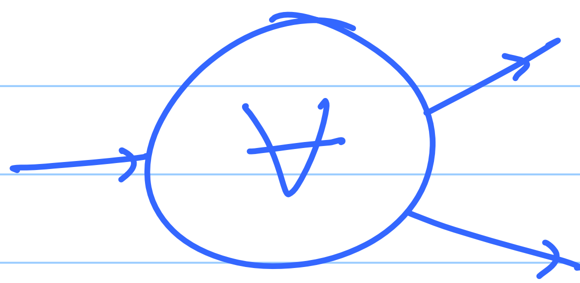
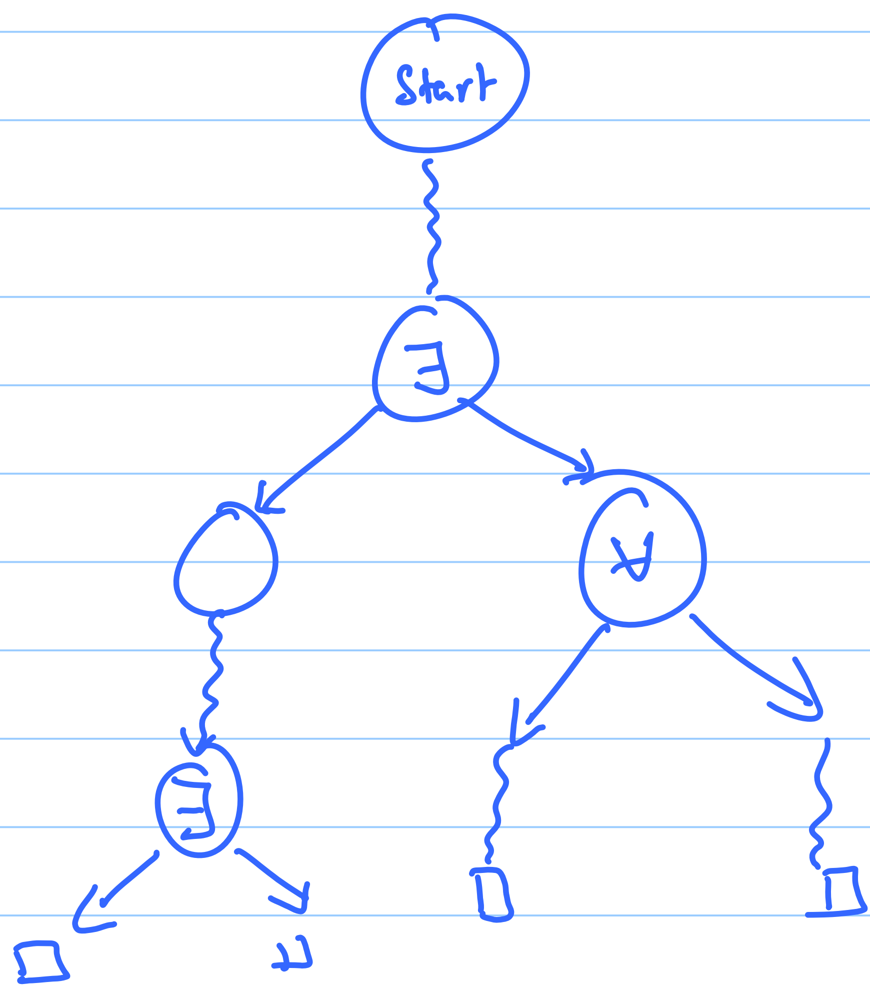


- Machine entering ∃ states accepts if one of the outgoing paths accepts.
- Machine entering ∀ state accepts if both outgoing paths accept.

(Natural extension of non-determinism)

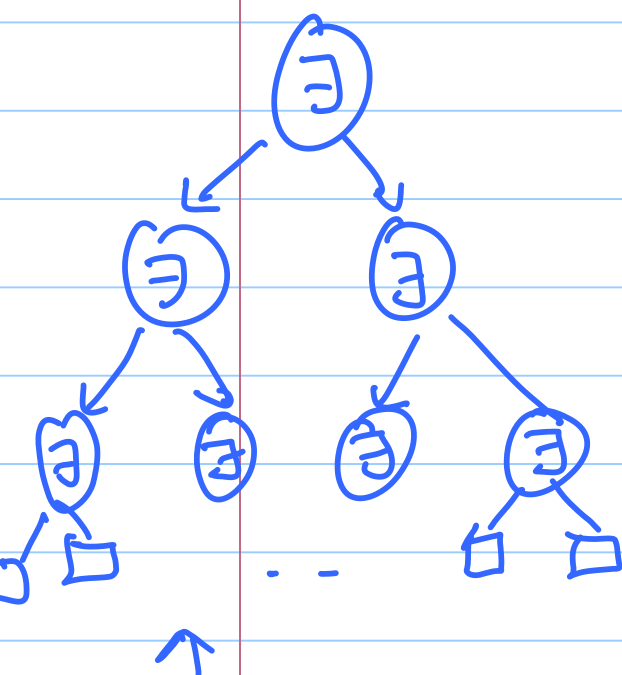
Computation of ATM

Represented by huge tree, with binary branching for ∃ & ∀ nodes.



Resources

- Time $\equiv$ Depth of tree
- Space $\equiv$ as usual
- Alternations $\equiv$ # switches from $\exists$ state to $\forall$ state in tree (max over paths) +1.



1 alternation

3 alternations.

Why study alternations ?

1. Models interesting problems

$$\text{MINDNF} = \{ (\phi, k) \mid \exists \psi \quad |\psi| \le k \text{ and } \forall x \ (\phi(x) = \psi(x)) \}$$

$$\text{TQBF} = \{ \phi \mid \exists x_1 \forall x_2 \ldots Q_n x_n \ \phi(x_1 \ldots x_n) = \text{true} \}.$$

2. Sheds light about TIME & SPACE

[FORTNOW]

Classical relationships

Theorem: $SPACE(S) \subseteq ATIME(S^2) \subseteq SPACE(S^2)$

Is this familiar? Replace ATIME with NSPACE .... !

Proof:

- $ATIME(S^2) \subseteq SPACE(S^2)$: Immediate since $ATIME(S^2)$ given by depth S^2 computation tree & can explore entire tree with space S^2.
- $SPACE(S) \subseteq ATIME(S^2)$

$REACH(C_1, C_2, 2^S)$

$$= \exists \underline{\underline{C_3}} \text{ s.t. } REACH(C_1, C_3, 2^{S-1})$$
$$\underline{\text{AND}}\ REACH(C_3, C_2, 2^{S-1})$$

takes S space to write ....

Theorem: $\mathrm{TIME}(2^s) \subseteq \mathrm{ASPACE}(O(s)) \subseteq \mathrm{TIME}(2^{O(s)})$

Proof: Second containment: # configurations $= 2^{O(s)}$.

- If graph on configurations = DAG then can determine whether start node accepts or not.
- if graph $\neq$ DAG, divide into strongly connected components;
  $\forall$ state in cycle is rejecting;
  $\exists$ state accepts if it leads to accepting state
  ⋮
  etc
- Can determine accept/reject nature of each configuration.

- First containment:

To determine if content of DTM T tableau in cell $(i,j) = \sigma$ can write ATM as follows:

$\text{CELL}(i,j,\sigma)$:

if $\exists\ \sigma_{-1}, \sigma_0, \sigma_1$ s.t.

$$\left\{ \forall b \in \{-1, 0, 1\} \quad \text{CELL}(i-b, j-1, \sigma_i) \right\}$$

AND $\begin{array}{|ccc|}\hline \sigma_1 & \sigma_2 & \sigma_3 \\ \hline & \multicolumn{1}{|c|}{\sigma_2} & \\ \cline{2-2} \end{array}$ valid for T

FORTNOW's THEOREM

Idea: • Assume $SAT \in LIN \triangleq \bigcup_c n \log^c n$

& $SAT \in L$

• Then alternations are not useful

(since $\underbrace{\exists \cdot LIN}_{SAT} \subseteq LIN$)

• But for small space computation, alternation is powerful; few alternation can reduce time significantly.

• $SAT \in L \Rightarrow$ all computation is small space.

• So all computation can be speeded up!

DETAILS : Assume SAT $\in$ SPACE$(c \cdot \log n)$

& SAT $\in$ TIME$(n^{1+\epsilon})$

- STEP 1 :

(1): $\text{TIME}(T(n)) \subseteq \text{SPACE}(c \cdot \log T(n))$

(needs a STRONG Cook's Theorem

SAT complete for NTIME(t(n)))

under LIN-reductions)

- STEP 2

(2): $\text{SPACE}(S) \subseteq \text{ATIME}[a, a \cdot 2^{S/a} \cdot S]$

alternations

time

(1) + (2) $\Rightarrow$ (3) :

$$\text{TIME}[T(n)] \subseteq \text{ATIME}\left[a, a(T(n))^{c/a} \cdot \log T(n)\right]$$

STEP 3

(4): $\mathrm{ATIME}[a, t] \subseteq \mathrm{TIME}\left[t^{(1+\epsilon)^a}\right]$

(Induction on a; Cook's Theorem)

(3) + (4): (5)

$$\mathrm{TIME}(T(n)) \subseteq \mathrm{TIME}\left(T(n)^{\frac{c}{a}\cdot(1+\epsilon)^a}\right)$$

contradiction if

$$\frac{c}{a}(1+\epsilon)^a < 1$$

$\Rightarrow$ $\forall\, c < \infty\ \exists\, \epsilon > 0$ s.t.

$\mathrm{SAT} \in \mathrm{SPACE}[c \cdot \log n]$.

$\Rightarrow$ $\mathrm{SAT} \notin \mathrm{TIME}(n^{1+\epsilon})$.