LECT 05

Note Title

TODAY

- Bounded depth circuits: AC^0
- Parity & AC^0 (mod "switching lemma")
- Proof of Switching Lemma.
  - Stage 1: → Case 1
    - → Case 2
  - Stage 2: → Case 1
    - → Case 2 → Part (a)
      - → Part (b)

AC^0: (more generally AC^k)

AC^k: Consist of poly-size circuits of depth $O(\log n)^k$ on $\{\infty\text{-AND}, \infty\text{-OR}, \text{NOT}\}$ gates

unbounded fanin.

[Not to be confused with NC^k

NC^k: poly size circuits depth $O(\log n)^k$ on $\{2\text{-AND}, 2\text{-OR}, \text{NOT}\}$

binary gates.

$NC^0 \subseteq AC^0 \subseteq NC^1 \subseteq AC^1 \subseteq \cdots\cdots$

Theorem: [Furst, Saxe, Sipser; Ajtai; Yao; Håstad; Razborov; Razborov-Smolensky]

$$\text{Parity} \notin AC^0$$

where $\text{Parity}(x_1 \ldots x_n) = \sum_{i=1}^{n} x_i \pmod 2$

Key Ingredient: Switching Lemma

Informally: Random assignment to most variables in DNF formula leaves very simple function on remaining

[Will elaborate later on random, simple, remaining]

[Mod Switching Lemma]:

- Set most variables of depth d circuit randomly
- Bottom 2 levels become so simple that circuit becomes depth $d-1$.

- Function (parity) on remaining variables remains complex (still a parity)
- Induction implies new circuit can't compute parity.

Definition: Random restriction ρ of $x_1 \dots x_n$ with parameter p does the following

$$\rho: \quad x_i \rightarrow \begin{cases} 0 & \text{w.p. } \frac{1-p}{2} \\ \star & \text{w.p. } p \\ 1 & \text{w.p. } \frac{1-p}{2} \end{cases}$$

Basic Switching Lemma : $\forall c_1 < \infty \;\exists C < \infty$ s.t.

- Let ρ be random restriction w. parameter $p = \frac{1}{\sqrt{n}}$;

- Let F be DNF formula of size $S \le n^{c_1}$

- Then $\Pr\left[F|_\rho \text{ depends on } > C \text{ variables} \right] \le \left(\frac{1}{n^{c_1}}\right)^2$

Notes: This version due to [F.S.S] ;

Improved versions due to [Håstad], [Razborov].

See survey by [Beame].

Proof that Parity $\notin AC^0$:

(assuming Basic S. L.)

- By induction assume $\forall c < \infty$ s.t. no depth d circuit of size 2^{n^c} computes parity.

- Now suppose n^{c_1} size depth $d+1$ circuit G computes parity.

- Apply r.r. ρ with parameter $p = \frac{1}{\sqrt{n}}$

- $\downarrow (1-2^{-\sqrt{n}})$ w.h.p. $\sqrt{n}$ variables remain;

- $G|_\rho$ computes parity of $\sqrt{n}$ variables;

[S.L.]

- w.p. $1 - \frac{1}{n^{2c_1}}$ any fixed gate at depth 2, computes fn. of C variables

- w.p. $1 - \frac{1}{n^{c_1}}$ all gates compute function of C variables

- $\Rightarrow G|_\rho \approx \tilde{G}$ of size $2^C \cdot n^{c_1}$ & depth d computing parity of $\sqrt{n}$ variables

- Contradiction to Induction if

$$2c_1 < c$$

$\boxtimes$

[Warning: Notes below somewhat incoherent/inconsistent]

Proof of Switching Lemma

- Do restriction in two stages

Stage 1 : x_i remains w.p. $\sqrt{p}$

Stage 2 : x_i remains w.p. $\sqrt{p}$

Claims :

Stage 1 : $\exists\, c$ depends only on $\log_n s$ s.t. each term has only c variables;

Stage 2 : $\exists\, C$ depending on c and $\log_n s$

s.t. DNF depends on only C variables.

[all w.p. $\geq 1 - \frac{1}{s^2}$]

Proof of Stage 1 claim: (relatively easy)

Case 1: $\wedge$ Fanout $\geq 4 \log s$

$$\Pr[\wedge \neq 0] \leq \left(\frac{1}{3}\right)^{4 \log s}$$

$$\lesssim \frac{1}{s^3}$$

$$\Pr[\exists \wedge \text{ with large fanout} \neq 0] \leq \frac{1}{s^2}$$

Case 2: $\wedge$ Fanout $\leq 4 \log s$

$$\Pr[i \text{ variables unrestricted}] \leq$$

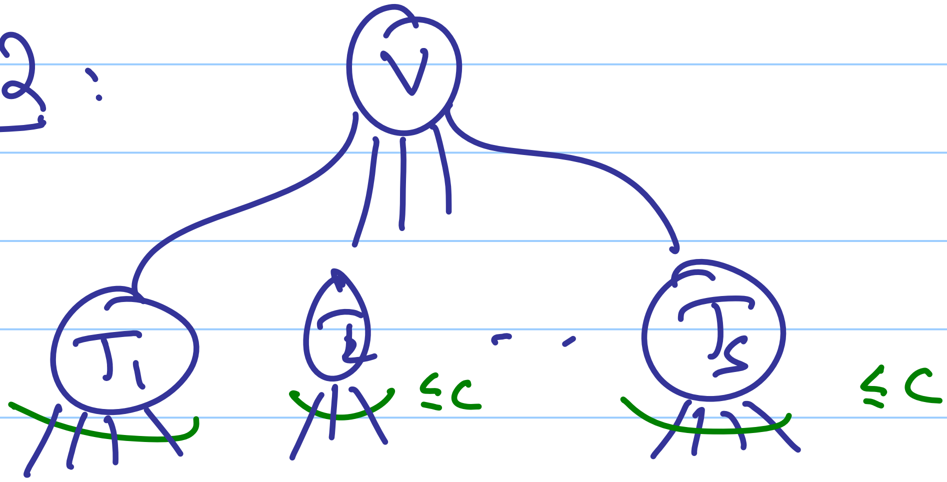
$$(4 \log s)^i (\sqrt{p})^i$$

$$\Pr[\exists \wedge \text{ gate with} \geq c \text{ unset variables}]$$

$$\leq s \cdot (4 \cdot \log s \cdot \sqrt{p})^c \quad \ldots$$

$\leq \frac{1}{s^2}$ [provided $c = \Omega(\log_n s)$]

Stage 2 :



Case 1 : # disjoint T_i's large $(\Omega(\log s))$

$$\Pr[T_i = 1] \geq \left(\frac{1}{3}\right)^c$$

$$\Pr[\exists\, T_i = 1] \geq 1 - \left(1 - \left(\frac{1}{3}\right)^c\right)^{\log s}$$

$$\geq 1 - \frac{1}{s^2}$$

(in this case we are done)

Case 2 : # disjoint T_i's $\leq$ $4 \log s$

Claim: $\exists$ $4c \log s$ variables H that hit every T_i

Proof: Select disjoint T_i's greedily.

When stop variables in selected T_i hit all T_i's. $\boxtimes$

Part 2ⓐ : First restrict variables of H;

As in Case 2 of Stage 1,

unset variables $\leq$ b'.

Part 2(b) : - Now restrict variables not in H_j & use induction on c.

- Let C' be s.t. w.h.p. $(c-1)$ DNF under random restriction depends on only C' variables

- Claim : w.h.p. c-DNF only depends on

$$C \leq b + 2^b \cdot C' \text{ variables.}$$

- Proof : - after restricting variables in H, enumerate each assignment to unset variables of H_j
- each leads to $(c-1)$-DNF $\Rightarrow$ depends on C' variables
- ours thus depends on $\leq b + 2^b \cdot C'$ variables. ☒