Surfaces

Polynomial Curves

- Bezier, Hermite, …
- Polynomial of degree *n*
- Example (cubic Bezier curve)

$$\mathbf{q}_b^T(u) = \begin{pmatrix} u^3 & u^2 & u^1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} (\mathbf{p}_1)_1 & (\mathbf{p}_1)_2 & (\mathbf{p}_1)_3 \\ (\mathbf{p}_2)_1 & (\mathbf{p}_2)_2 & (\mathbf{p}_2)_3 \\ (\mathbf{p}_3)_1 & (\mathbf{p}_3)_2 & (\mathbf{p}_3)_3 \\ (\mathbf{p}_4)_1 & (\mathbf{p}_4)_2 & (\mathbf{p}_4)_3 \end{pmatrix}$$

- Or equivalently (see Buss, VII.1) $q_j(u) = \sum_i B_i(u)(\mathbf{p}_i)_j$

Splines: Piecewise Curves

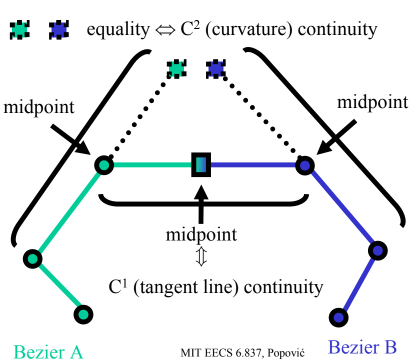
- Catmull-Rom, B-spline, …
- Interpolate or approximate control points by assembling polynomial curves.
- Example (Catmull-Rom spline):
  - Piecewise cubic
  - Interpolates input points $p_1, p_2, \ldots, p_n$.
  - Unlike Bezier, there can be more than four points
  - C^1 (tangent line) continuity
  - More details (Buss, VII.15.1)

Example: Uniform B-spline

- Piecewise cubic
- Approximates input points.
- Unlike Bezier, there can be more than four points
- Unlike Camtull-Rom, it does not interpolate
- C^2 (curvature) continuity

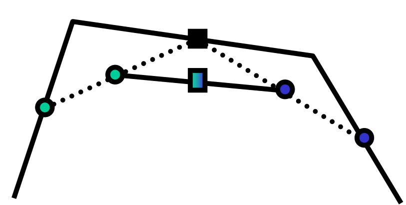
- Algebraic construction (Buss, VIII.1)

B-spline: Geometric Construction



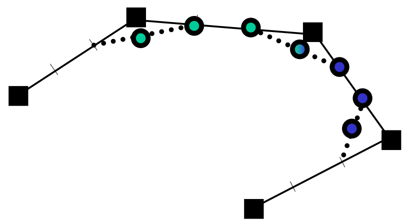
B-spline: Proof

- Proof of Bezier C^2 continuity via Subdivision/de Casteljau construction



B-spline

- B-spline control points are black squares
- Control points for two Bezier curves are circles



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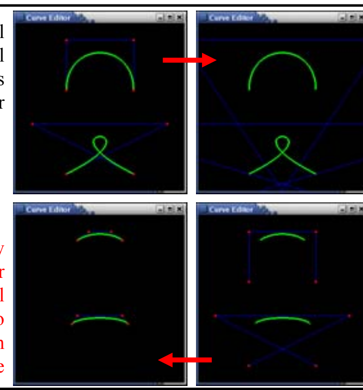
Converting between Bézier & BSpline

original control points as Bézier

new BSpline control points to match Bézier



new Bézier control points to match BSpline

original control points as BSpline

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Conversion: Analytic Form

- From B-spline control points to Bezier control points.
- Slide $\mathbf{G}_s$ as a window of four B-spline points to obtain control points for each Bezier curve.

$$\mathbf{G}_b = \frac{1}{6}\begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 1 \end{pmatrix}\mathbf{G}_s$$

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Surfaces

- Swept Surfaces
  - Surfaces of revolution
  - General surfaces
- Tensor-Product Surfaces

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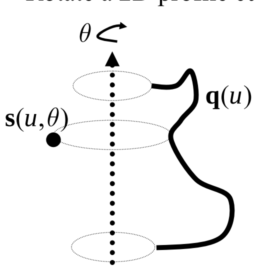
Surfaces of Revolution

- Rotate a 2D profile curve around and axis.



$$\mathbf{s}(u,\theta) = \mathbf{R}(\theta)\mathbf{q}(u)$$

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Surface Normal Vectors

- Normal vectors are needed for shading
  - Normal vector perpendicular to tangent plane
  - Tangent plane spanned by partial derivatives
  - So:

$$\vec{\mathbf{n}}(u,\theta) = \text{normalize}\left(\frac{\partial \mathbf{s}(u,v)}{\partial u} \times \frac{\partial \mathbf{s}(u,v)}{\partial v}\right)$$

  - In the special case of surface of revolution:

$$\vec{\mathbf{n}}(u,\theta) = \mathbf{R}(\theta)\ \text{normal}(\mathbf{q}(u))$$

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General Sweep Surfaces

- Trace out surface by moving a profile curve along a trajectory.
- Orientation options:
  - Align profile curve with an axis.
  - Align profile curve with a Frenet frame.
  - Analytic form uses a matrix and a trajectory to transform the profile curve:

$$\mathbf{s}(u,v) = \mathbf{M}(\mathbf{c}(u))\mathbf{q}(u)$$

Bezier Tensor Products

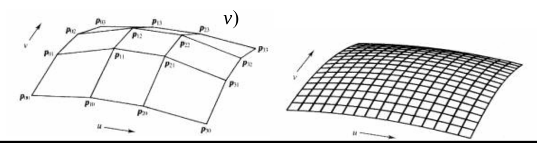
- Use a 4x4 grid of control $\mathbf{p}_{ij}$ points to build a surface:
  - Use the four rows as control points for four Bezier curves: $\mathbf{q}_0(u)$, $\mathbf{q}_1(u)$, $\mathbf{q}_2(u)$, $\mathbf{q}_3(u)$
  - Define a point on the surface $\boldsymbol{s}(u,v)$ by evaluating another Bezier curve (for parameter v) using the four control points defined by for row Bezier curves (for some value u).

Bezier Tensor Products

Notation: $\mathbf{CB}(P_1, P_2, P_3, P_4, \alpha)$ is Bézier curve with control points P_i evaluated at α

Define "Tensor-product" Bézier surface

$$\mathbf{s}(u,v) = \mathrm{CB}(\ldots \\ \mathrm{CB}(\mathbf{p}_{00}, \mathbf{p}_{01}, \mathbf{p}_{02}, \mathbf{p}_{03}, u), \\ \mathrm{CB}(\mathbf{p}_{10}, \mathbf{p}_{11}, \mathbf{p}_{12}, \mathbf{p}_{13}, u), \\ \mathrm{CB}(\mathbf{p}_{20}, \mathbf{p}_{21}, \mathbf{p}_{22}, \mathbf{p}_{23}, u), \\ \mathrm{CB}(\mathbf{p}_{30}, \mathbf{p}_{31}, \mathbf{p}_{32}, \mathbf{p}_{33}, u), \\ v)$$

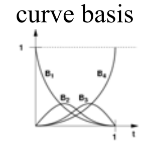


Basis Form

- Derivation (Buss VII.10)

$$\mathbf{s}(u,v) = \sum_{i=0}^{3} \sum_{j=0}^{3} B_i(u) B_j(v) \mathbf{p}_{ij}$$

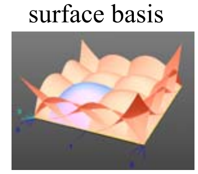
curve basis



- Tensor-product basis by definition of the tensor product:

$$B_{ij}(u,v) = B_i(u) B_j(v)$$

surface basis



Matrix Form

- First coordinate only:

$$(\mathbf{s}(u,v))_1 = \begin{pmatrix} B_0(u) & B_1(u) & B_2(u) & B_3(u) \end{pmatrix} \begin{pmatrix} (\mathbf{p}_{00})_1 & \cdots & (\mathbf{p}_{03})_1 \\ \vdots & \ddots & \vdots \\ (\mathbf{p}_{30})_1 & \cdots & (\mathbf{p}_{33})_1 \end{pmatrix} \begin{pmatrix} B_0(v) \\ B_1(v) \\ B_2(v) \\ B_3(v) \end{pmatrix}$$

$$= \begin{pmatrix} u^3 & u^2 & u^1 & 1 \end{pmatrix} \mathbf{B}_{\text{bezier}} \mathbf{P}_1 \mathbf{B}_{\text{bezier}}^{\mathrm{T}} \begin{pmatrix} v^3 \\ v^2 \\ v \\ 1 \end{pmatrix}$$

Tensor-Product B-splines

- Use a mxm grid of control points.
- Composed of of many Bezier surface patches. The (k,l) patch:

$$\mathbf{s}_{kl}(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{m} N_i(u) N_j(v) \mathbf{p}_{ij}$$

What You Should Know To Do

- List definitions and properties of polynomial curves, splines, B-splines, surfaces of revolution, generalized sweep surfaces, tensor-product surfaces.
- List definitions of C and G continuity and recognize differences visually.
- Derive analytic expressions for polynomial curves and spline from constraints indicating locations, tangents, and continuity.
- Evaluate Bezier and B-splines with geometric construction.
- Display polynomial curves and splines using line segments.