# 6.825 Exercise Solutions, Decision Theory

1 Decision Theory I

Dr. No has a patient who is very sick. Without further treatment, this patient will die in about 3 months. The only treatment alternative is a risky operation. The patient is expected to live about 1 year if he survives the operation; however, the probability that the patient will not survive the operation is 0.3.

1. Draw a decision tree for this simple decision problem. Show all the probabilities and outcome values.
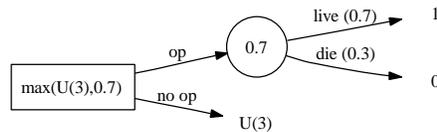   **Solution:**
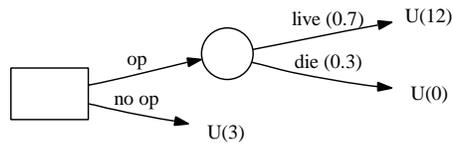
2. Let $U(x)$ denote the patient's utility function, where x is the number of months to live. Assuming that $U(12) = 1.0$ and $U(0) = 0$, how low can the patient's utility for living 3 months be and still have the operation be preferred?

   For the rest of the problem, assume that $U(3) = 0.8$.
   Solution: The operation would be preferred as long as $U(3) < 0.7$.

3. Dr. No finds out that there is a less risky test procedure that will provide uncertain information that predicts whether or not the patient will survive the operation. When this test is positive, the probability that the patient will survive the operation is increased. The test has the following characteristics:

   - True-positive rate: The probability that the results of this test will be positive if the patient will survive the operation is 0.90.
   - False-positive rate: The probability that the results of this test will be positive if the patient will not survive the operation is 0.10.

   What is the patient's probability of surviving the operation if the test is positive?
   Solution:

$\Pr(survive|pos) = \frac{\Pr(pos|survive)\Pr(survive)}{\Pr(pos)} = \frac{\Pr(pos|survive)\Pr(survive)}{\Pr(pos|survive)\Pr(survive)+\Pr(pos|\neg survive)\Pr(\neg survive)} = \frac{0.9\cdot0.7}{0.66} = 0.9545.$

4. Assuming the patient has the test done, at no cost, and the result is positive, should Dr. No perform the operation?
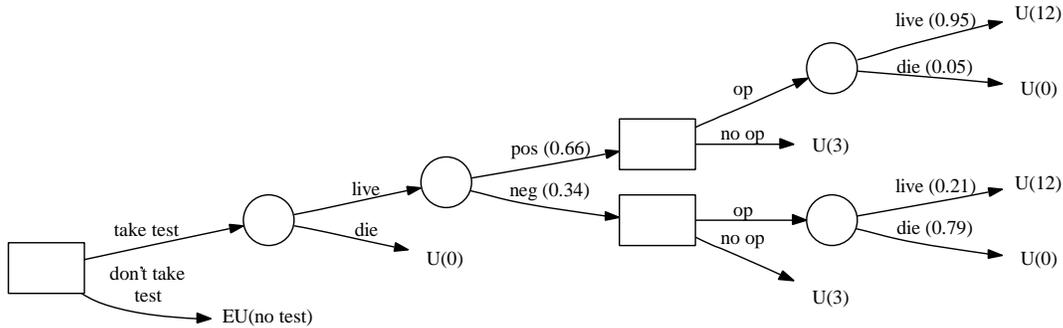   **Solution:** Yes. $EU(op) > 0.8$.

5. It turns out that the test may have some fatal complications, i.e., the patient may die during the test. Draw a decision tree showing all the options and consequences of Dr. No's problem.
   Solution:
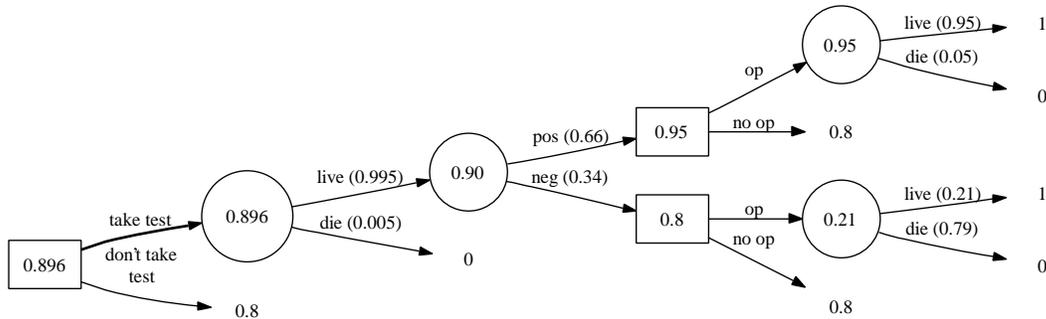   For this problem, we need to calculate $\Pr(survive|neg)$ as above:
   $\Pr(survive|neg) = 0.2059$.
   Then we have the following decision tree:

6. Suppose that the probability of death during the test is 0.005 for the patient. Should Dr. No advise the patient to have the test prior to deciding on the operation?
   Solution: Yes, the test should be taken. The evaluated decision tree is as follows:

2 Decision Theory II

You go to the racetrack. You can:

- Decline to place any bets at all.

- Bet on Belle. It costs $1 to place a bet; you will be paid $2 if she wins (for a net profit of $1).

- Bet on Jeb. It costs $1 to place a bet; you will be paid $11 if he wins (for a net profit of $10).

You believe that Belle has probability 0.7 of winning and that Jeb has probability 0.1 of winning.

1. Your goal is to maximize the expected value of your actions. What, if any, bet should you place, and what is your expected value? Draw the decision tree that supports your conclusion. Assume that you are risk-neutral.
   Solution: You should bet on Belle. The expected utility is $U(\$0.40)$.

2. Someone comes and offers you gambler's anti-insurance. If you agree to it,

   - they pay you $2 up front
   - you agree to pay them 50% of any winnings (that is, $.50 if Belle wins, and $5 if Jeb wins).

   How would it affect the expected value of each of your courses of action? What would be the best action to take now, again assuming risk-neutrality? Draw the new decision tree.
   Solution: You should take the $2 insurance and bet on Belle. The expected utility is $U(\$2.05)$ (see figure 1).
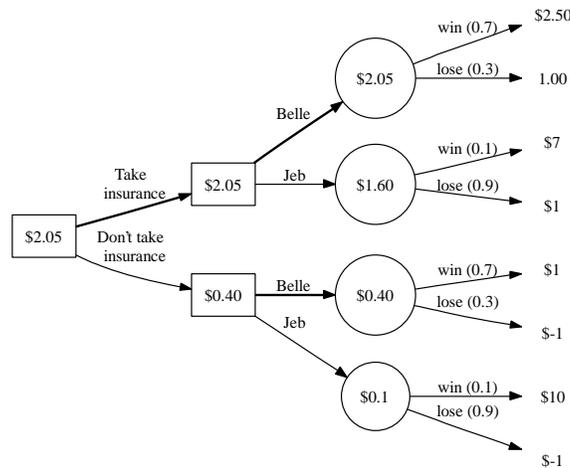

Figure 1: Decision tree for the betting problem

3 Decision Theory III

You're an olympic skier. In practice today, you fell down and hurt your ankle. Based on the results of an x-ray, the doctor thinks that it's broken with probability 0.2. So, the question is, should you ski in the race tomorrow?

If you ski, you think you'll win with probability 0.1. If your leg is broken and you ski on it, then you'll damage it further. So, your utilities are as follows: if you win the race and your leg isn't broken, +100; if you win and your leg is broken, +50; if you lose and your leg isn't broken 0; if you lose and your leg is broken -50.

If you don't ski, then if your leg is broken your utility is -10, and if it isn't, it's 0.

1. Draw the decision tree for this problem.

2. Evaluate the tree, indicating the best action choice and its expected utility.

Solution:
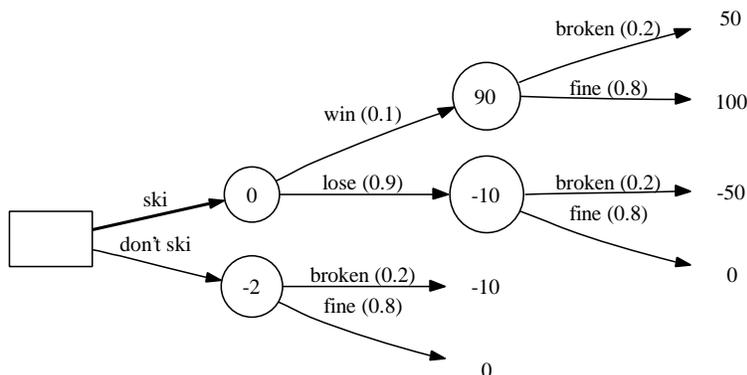U(ski) = 0 and U(not ski) = -2, so we ski (see figure 2).

Figure 2: Decision tree for the skier's choice.

You might be able to gather some more information about the state of your leg by having more tests. You might be able to gather more information about whether you'll win the race by talking to your coach or the TV sports commentators.

3. Compute the expected value of perfect information about the state of your leg.

Solution:
Given perfect information about the leg, we have the tree in figure 3, so the expected value of the information is $E(U_{info}) - E(U_{noinfo}) = 6 - 0 = 6$.

4. Compute the expected value of perfect information about whether you'll win the race.

Solution:
Given perfect information about winning, we have the tree in figure 4, so the expected value of the information is $E(U_{info}) - E(U_{noinfo}) = 7.2 - 0 = 7.2$.

In the original statement of the problem, the probability that your leg is broken and the probability that you'll win the race are independent. That's a pretty unreasonable assumption.

6. Is it possible to use a decision tree in the case that the probability that you'll win the race depends on whether your leg is broken?
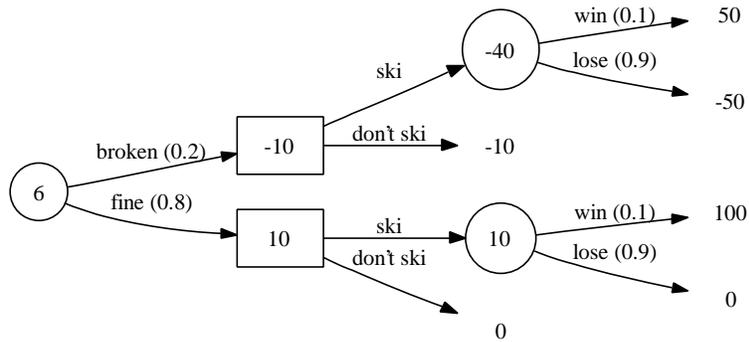
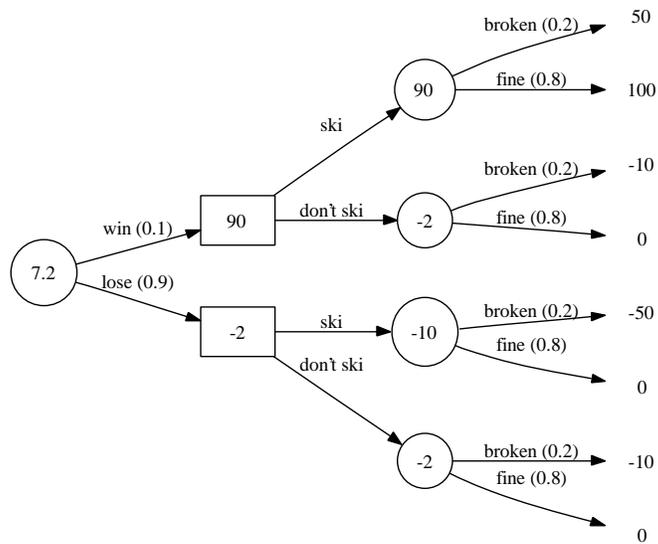Figure 3: Decision tree given perfect information about the leg.

Figure 4: Decision tree given perfect information about winning.

Solution:
Yes. Just put the win branch after the broken branch and use the conditional probabilities for the given state of the leg.