6.5440: Algorithmic Lower Bounds, Fall 2023

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Problem Set 9 Solution

Due: Monday, November 13, 2023 at noon

Problem 9.1 [Edge Matching ASP-completeness].

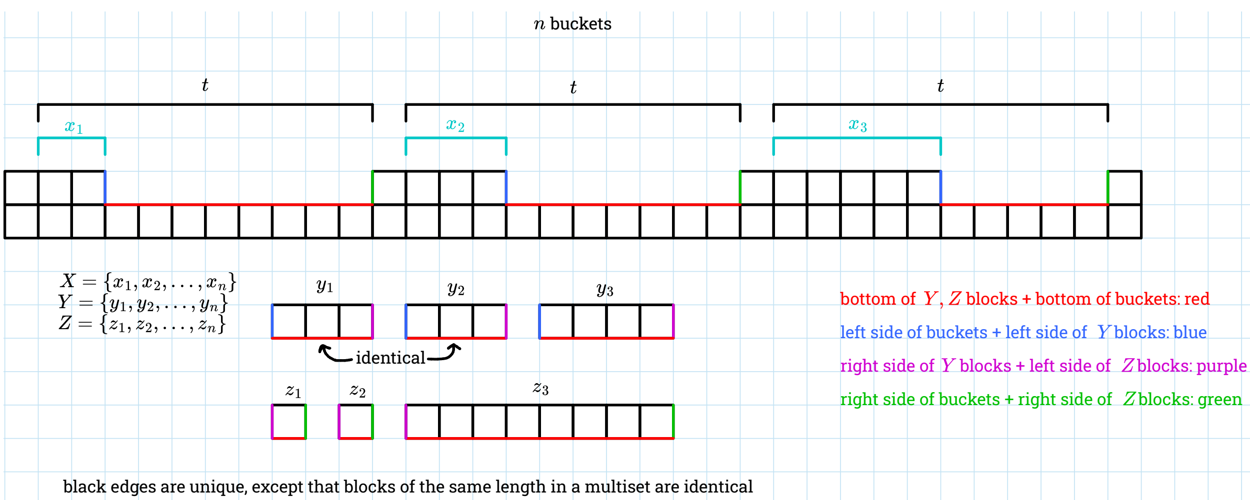
Recall from Problem Set 2 (solutions) that $2 \times n$ *edge matching, forbidding reflections and rotations* is NP-complete.

Prove that this problem is in fact ASP-complete, by giving a parsimonious reduction from an ASP-complete problem.

You must include a drawing or diagram in your submission.

Hint: Use the fact that Numerical 3-Dimensional Matching is ASP-complete, as proved in "Path Puzzles: Discrete Tomography with a Path Constraint is Hard" by Bosboom, Demaine, Demaine, Hesterberg, Kimball, and Kopinsky (2020).

Solution:



We give a parsimonious reduction from Numerical 3-Dimensional Matching to $2 \times n$ edge matching without rotations and reflections.

Let the Numerical 3DM instance be X, Y, Z, where X is a set of cardinality n and Y, Z are multisets each of cardinality n. (We are guaranteed that X is a set by Theorem 2.4 in the Path Puzzles paper.) Let $t = \frac{\sum X + \sum Y + \sum Z}{n}$ be the target sum. As in the solution to Problem Set 2, we make a single large frame which forms n buckets, into each of which will be placed three small blocks. As before, the frame is constructed using unique colors on each internal edge and on every edge which should touch the exterior, ensuring that it can be assembled in the desired way.

Since we are now reducing from Numerical 3DM, we'll use colors on the sides of the blocks to ensure each bucket has one element from each of X, Y, Z. These colors will also fix the order of blocks within a bucket. We also fix the particular element of X assigned to each bucket, to prevent additional solutions from arising by permuting the triples of blocks among the buckets. We can do this by just embedding the blocks corresponding to elements of X directly into the large frame, as shown in the figure.

Finally, we'll specify how to build the tilesets corresponding to numbers in Y and Z. For each number y appearing with multiplicity m in Y, we construct a $1 \times y$ block. The left, bottom, and right sides are blue, red, and purple respectively according to the figure. We choose fresh (i.e. previously unused) colors to connect the y tiles together to form the block, as well as fresh colors for the top. Then we duplicate these y tiles m times to obtain ym tiles in total, which are forced to be assembled into m identical $1 \times y$ blocks. Finally, we also do the same for each number in Z, except that the left and right sides are purple and green according to the figure.

Consider the function f which maps solutions of the constructed edge-matching instance to solutions to the original Numerical 3DM instance, defined by

$$f(T) = \{(x, y, z) \in X \times Y \times Z \mid \text{A bucket containing blocks of size } x, y, z \text{ in sequence appears in tiling } T\}.$$

We can define an inverse f^{-1} as follows. Let $S \subset X \times Y \times Z$ be a solution to the Numerical 3DM instance. Order the tuples of S such that the first element of the i'th tuple is equal to the length of the X block embedded into the i'th bucket in the edge-matching instance; because X is a set, this ordering is well-defined. Then assemble the Y and Z blocks corresponding to the i'th tuple into the i'th bucket.

It can be checked that f^{-1} is indeed an inverse to f, and so the reduction is parsimonious. The critical point is that since the m blocks corresponding to a Y or Z element with multiplicity m are identical, the same tiling is obtained no matter which of the m blocks is placed.