6.5440: Algorithmic Lower Bounds, Fall 2023

Prof. Erik Demaine, Josh Brunner, Lily Chung, Jenny Diomidova

Problem Set 8 Solution

Due: Monday, November 6, 2023 at noon

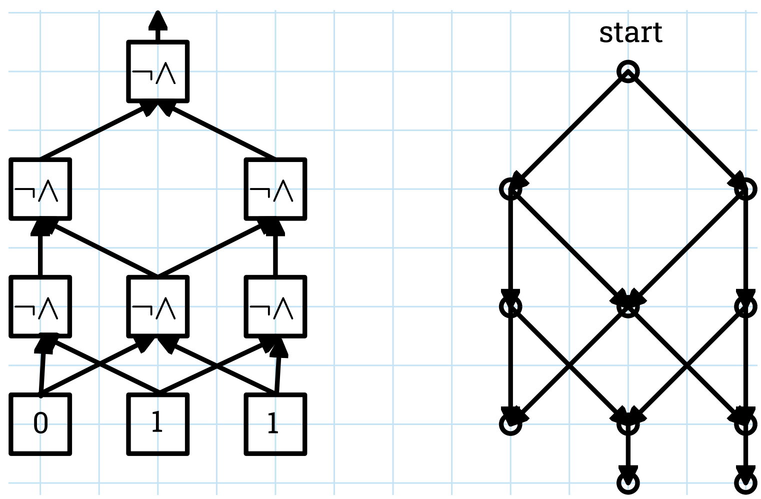
Problem 8.1 [Acyclic Geography].

In ***directed vertex geography*** (introduced in Lecture 13), players take turns moving a single token along the edges of a directed graph, without repeating previously visited vertices. The loser is the first player who is unable to move. In general directed graphs, determining the winner is PSPACE-complete.

Prove that directed vertex geography is P-hard when played on a ***directed acyclic graph***, that is, a graph with no directed cycles. (In fact, this problem is P-complete.) Remember that P-hardness reductions need to be NC algorithms. (Alternatively, it suffices to guarantee your reduction uses logarithmic space.)

You must include a drawing or diagram in your submission.

Solution:



Call a vertex is ***winning*** if the player to move wins when the token is at that vertex. It is straightforward to see that a vertex v with outgoing edges to vertices w_1 and w_2 is winning if and only if w_1 and w_2 are not both winning. Using this fact, we can reduce from NAND-CVP by transforming every gate G_i into a vertex v_i, and every connection between the output of G_i to the input of a NAND gate G_j into an edge $v_j \to v_i$. For each input gate G_i we add an extra vertex v_i' and an edge $v_i \to v_i'$ if G_i has value 1.

This reduction can be computed in logarithmic space, hence in NC. Determining whether an edge $v_j \to v_i$ exists requires only checking whether G_i is an input to G_j. Similarly for v_i corresponding to input gates, determining whether there is an edge $v_i \to v_i'$ (and whether v_k' exists) requires only reading the input value of G_i.