6.5440: ALGORITHMIC LOWER BOUNDS, FALL 2023
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# Problem Set 5 Solution

*Due: Tuesday, October 10, 2023 at noon*

**Problem 5.1 [$1 \times n$ edge-matching].**

Recall that an (unsigned) ***edge-matching puzzle*** consists of $mn$ tiles, where each tile is a unit square whose sides are each labeled with a color. The goal is to place the $mn$ given tiles into an $m \times n$ rectangle such that tiles match in color on shared edges.

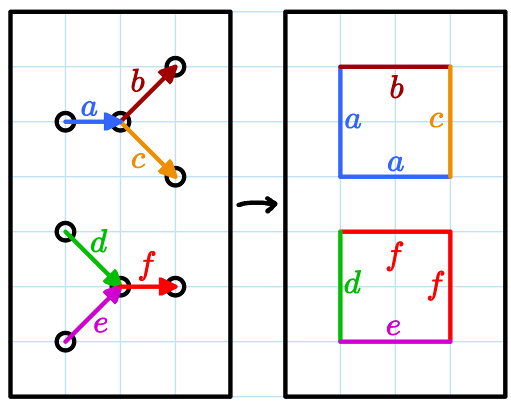
Give a reduction from Hamiltonicity in Directed 3-Regular Graphs to show that edge-matching puzzles are NP-hard even when the following two conditions hold simultaneously:

- $m = 1$, i.e., the puzzle is $1 \times n$; and
- tiles can be freely rotated, reflected, and translated.

By contrast, in Problem Set 2, you were asked to prove NP-hardness of $2 \times n$ edge matching ($m = 2$) when tiles *cannot* be rotated or flipped.

**You must include a drawing or diagram in your submission.**

**Solution:** We reduce from the Hamiltonian path problem in directed 3-regular graphs. Let $G$ be a directed 3-regular graph on $n$ vertices. We will have one tile for each vertex of $G$, and one color for each edge of $G$. For each vertex with incoming edge $a$ and outgoing edges $b, c$ we create a tile with edges colored (in rotational order) $a, a, b, c$. For each vertex with incoming edges $d, e$ and outgoing edges $f$, we create a tile with edges colored $d, e, f, f$.




This tile set can be computed from $G$ in polynomial time. We claim that there exists a Hamiltonian path in $G$ if and only if the tile set admits a $1 \times n$ edge matching.

Suppose there exists a Hamiltonian path $v_1, v_2, \ldots, v_n$. Then we can place the corresponding tiles in the same order, and rotate them so that the color of the border between tiles $v_i$ and $v_{i+1}$ corresponds to the edge $(v_i, v_{i+1})$. This can always be done because for any pair of incoming and outgoing edge through $v_i$, there is an orientation of the corresponding tile such that the incoming edge's color is on the left and the outgoing edge's color is on the right. Thus we obtain a valid edge matching.

Now suppose instead that there exists a $1 \times n$ edge matching. Number the vertices $v_1, v_2, \ldots, v_n$ according to their order of appearance in the matching. Every pair of consecutive vertices $v_i, v_{i+1}$ shares an edge of $G$: it is the edge corresponding to the color of the shared border between the tiles. If the edge in question is $(v_i, v_{i+1})$, we call it ***forwards***; if it is $(v_{i+1}, v_i)$, we call it ***backwards***.

Now consider three consecutive vertices $v_i, v_{i+1}, v_{i+2}$ and the two edges corresponding to the borders between them. By considering $v_{i+1}$'s tile, we can see that the two edges are either both forwards or both backwards. Therefore the edges in the edge matching are either all forwards or all backwards. So they form a directed Hamiltonian path.