Problem Set 4 Solution

Due: Monday, October 2, 2023 at noon

Problem 4.1 [Planar Positive NAE EU3SAT].
Let ϕ be an instance of Not-All-Equal SAT satisfying the following properties:

Positive: Negations are not allowed; ϕ contains only positive literals.

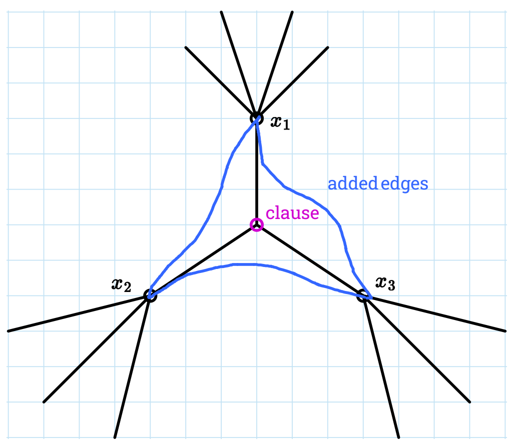
EU3: Every clause contains exactly three distinct variables.

Planar: The following graph G is planar: the graph with a vertex for each variable and each clause, and edges connecting vertices to the clauses they appear in.

Prove that ϕ is always satisfiable. **You must include a drawing or diagram in your submission.**

Hint: Use the Four-Color Theorem, which states that the vertices of any planar graph can be colored with four colors so that adjacent vertices have different colors. You may need to modify graph G (e.g., by adding extra edges) before coloring it in order to make the coloring more useful.

Solution:



Let G' be the same graph as G, except that for each clause (x_1, x_2, x_3), we add the edges (x_1, x_2), (x_2, x_3), and (x_3, x_1). It can be seen from the figure that G' is still planar. Let $f : V(G') \to \{0, 1, 2, 3\}$ be a 4-coloring of G', and define

$$\sigma(x) = (f(x) \in \{0, 1\})$$

We claim that σ is a satisfying assignment of ϕ. Consider any clause $C = (x_1, x_2, x_3)$. Because the edges $(x_1, x_2), (x_2, x_3), (x_3, x_1)$ are present in G', it must be that all of $\{f(x_1), f(x_2), f(x_3)\}$ are distinct. Thus at least one of $\{f(x_1), f(x_2), f(x_3)\}$ is in $\{0, 1\}$ and one of them is in $\{2, 3\}$, so $\{\sigma(x_1), \sigma(x_2), \sigma(x_3)\}$ are not all equal to each other. Therefore C is satisfied.

This idea is from "Planar 3-SAT with a Clause/Variable Cycle" by Pilz (2008), where it is used to prove Theorem 12, which states that a Planar SAT instance is always satisfiable if every clause has either at least three positive literals or at least three negative literals. In general it applies to any Boolean Planar CSP instance in which each clause has three variables x_1, x_2, x_3 such that, if x_1, x_2, x_3 are not all equal, then the clause is satisfied.