6.5440: Algorithmic Lower Bounds, Fall 2023

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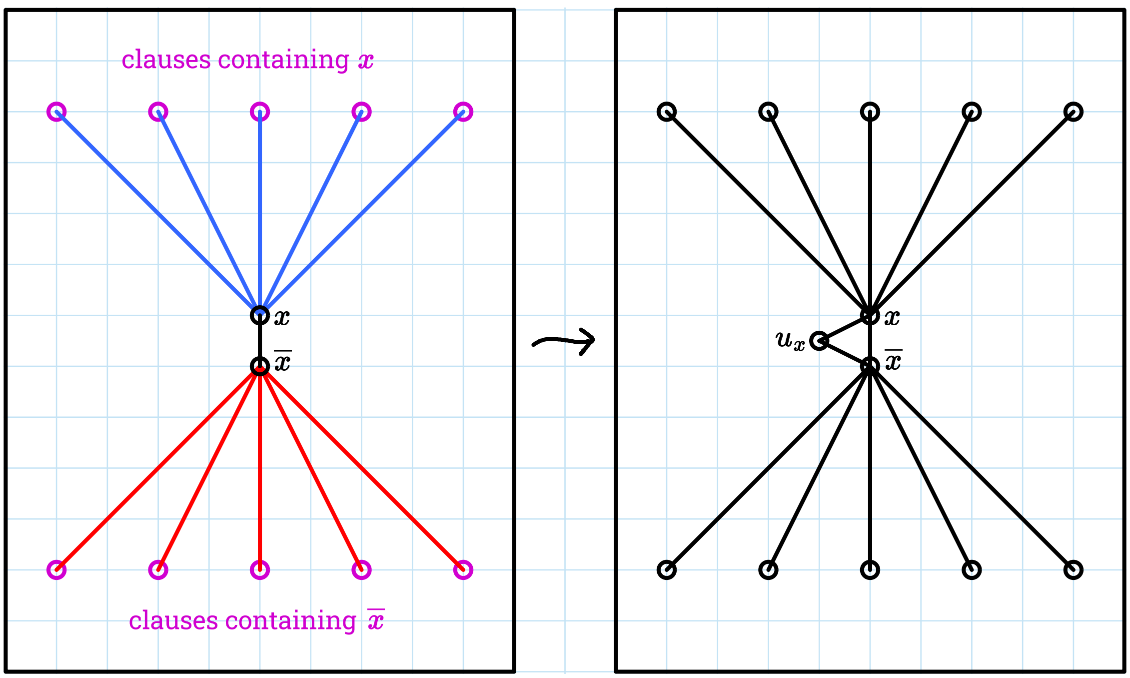
Problem Set 3 Solution

Due: Monday, September 25, 2023 at noon

Problem 3.1 [Planar dominating set]. Given an undirected graph $G = (V, E)$, a ***dominating set*** is a subset of vertices $S \subseteq V$ such that every other vertex $v \in V - S$ is adjacent to at least one vertex in S. The Planar Dominating Set problem is to decide, given a planar graph G and a positive integer k, whether G has a dominating set of size $\le k$.

Planar Dominating Set is contained in NP. Show that it is NP-hard.

Solution:



We reduce from Strongly Planar 3SAT mentioned in Lecture 5: deciding satisfiability of a 3CNF formula φ on n variables such that the graph H_φ is planar, where H_φ consists of a vertex for every *literal* and clause, together with edges connecting clauses to their literals and an edge connecting the two literals of each variable.

Our instance G of Planar Dominating Set will consist of the following:

- For each variable x: three vertices $x, \overline{x}, u_x$ and all three edges between these vertices, forming a triangle.
- For each clause C: a vertex C, together with three edges connecting C to vertices corresponding to its literals. For instance, if $C = (x \vee \overline{y} \vee z)$, then there are edges (C, x), $(C, \overline{y})$, (C, z).

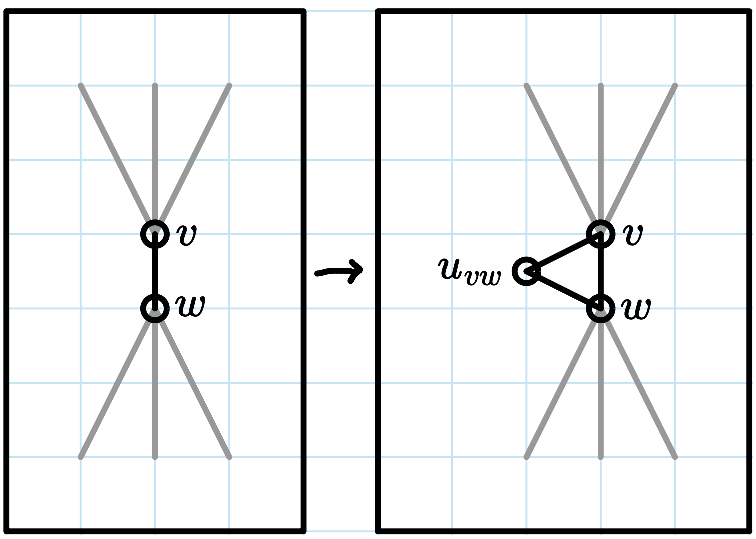
The resulting graph G is planar: indeed, G is the same as H_φ except for the vertices u_x and their edges, which can be added without violating planarity. It is clear that G can be computed in polynomial time. We claim that G has a dominating set of size $\le n$ if and only if φ is satisfiable.

First, suppose σ is a satisfying assignment for φ. Let S be the set of literals assigned true by σ; clearly $|S| = n$. Then S is a dominating set: every vertex in $\{x, \overline{x}, u_x\}$ is clearly adjacent to whichever of x and $\overline{x}$ is in S, while every vertex C is adjacent to a vertex in S by the assumption that σ satisfies φ. So S is a solution to the Planar Dominating Set instance (G, n).

Now, suppose that S is a dominating set with $|S| \leq n$. Because each u_x is adjacent only to x and $\overline{x}$ it follows that S must include at least one of $\{x, \overline{x}, u_x\}$ for each variable x; since $|S| \leq n$ it in fact includes exactly one of them. We claim that σ satisfies φ, where $\sigma(x) = (x \in S)$. Since S is dominating it contains at least one literal in every C; and any literal in S is assigned true by σ because S doesn't contain both x and $\overline{x}$ simultaneously.

So (H, n) is in Planar Dominating Set if and only if φ is in Strongly Planar 3SAT, so we have demonstrated a reduction from Strongly Planar 3SAT to Planar Dominating Set. Thus Planar Dominating Set is NP-hard.

Solution:



We reduce from Planar Vertex Cover. Given an instance (G, k) of Planar Vertex Cover, we produce in polynomial time an instance (H, k) of Planar Dominating Set as follows.

Without loss of generality we can assume G has no isolated vertices, since removing them does not change the vertex cover instance. We create H identical to G, except that for each edge (v, w) of G we add a new vertex u_{vw} and edges $(w, u_{vw}), (v, u_{vw})$. These can be added without violating planarity. We claim that H has a dominating set of size $\leq k$ if and only if G has a vertex cover of size $\leq k$.

In one direction, any vertex cover of G is a dominating set of H (since G has no isolated vertex). In the other direction, let S be a dominating set of H. Define S' identical to S except that any vertex $u_{vw} \in S$ is replaced with v instead. Clearly $|S'| \leq |S|$. Then S' is a vertex cover of G, since for every edge (v, w) at least one of v or w is in S' in order to neighbor u_{vw}.

Thus (H, k) is in Planar Dominating Set if and only if (G, k) is in Planar Vertex Cover, so Planar Dominating Set is NP-hard.