6.5440: Algorithmic Lower Bounds, Fall 2023
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Problem Set 10 Solution

Due: Monday, November 20, 2023 at noon

Problem 10.1 [Twenty Questions].

Let U be a finite set ("the universe"). For two distinct elements $x, y \in U$, a set $T \subset U$ ***separates*** x from y if either:

- $x \in T$ and $y \notin T$; or
- $x \notin T$ and $y \in T$.

A collection $S' \subseteq 2^U$ of subsets of U is ***discerning*** if, for every two distinct elements $x, y \in U$, there exists some $T \in S'$ that separates x from y.

The Minimum Questions problem is, given a set U and a collection $S \subseteq 2^U$ of subsets of U, to find the smallest possible discerning collection $S' \subset S$; that is, the discerning collection consisting of the smallest possible number of subsets.

Prove that there is a ***log-approximation-preserving*** reduction f, g from Minimum Set Cover to Minimum Questions, in the following sense:

- f is a function from Set Cover instances to Questions instances.
- g is a function from solutions of $f(x)$ to solutions of x.
- f, g are computable in polynomial-time.
- For all functions $\alpha \in O(\log n)$ there exists a function $\beta \in O(\log n)$ satisfying the following. For all instances x and all solutions y' of $f(x)$, if y' is an α-approximation for $f(x)$ i.e. $\frac{\text{cost}(y')}{OPT(f(x))} \leq \alpha(|f(x)|)$, then $g(y')$ is a β-approximation for x i.e. $\frac{\text{cost}(g(y'))}{OPT(x)} \leq \beta(|x|)$.

Such a reduction implies that, if you have an $O(\log n)$-approximation algorithm for Minimum Questions, then you can obtain an $O(\log n)$-approximation algorithm for Minimum Set Cover, which is known to be impossible unless P = NP.

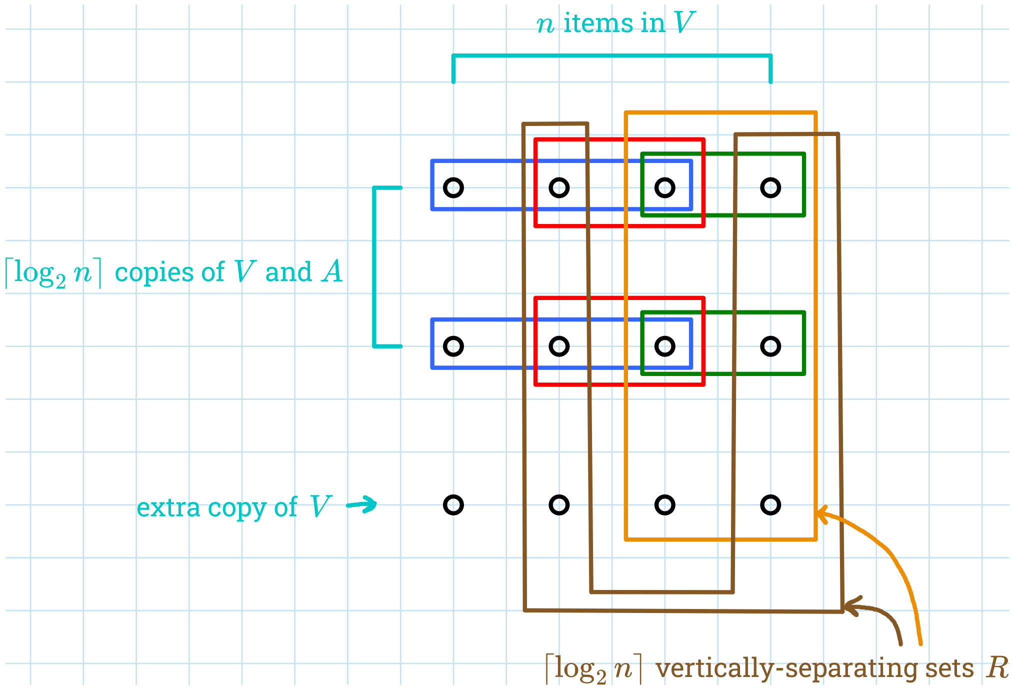
You must include a drawing or diagram in your submission.

Hint 1: There is a reduction from Set Cover to the decision version Questions as follows. Suppose the Set Cover instance has elements $V = \{v_0, \ldots, v_{n-1}\}$ and sets $A \subseteq 2^V$. Then the reduction to Questions is defined by:

$$\begin{aligned}
U &= V \times \{0, 1\}, \\
S &= S_1 \cup S_2, \\
S_1 &= \{T \times \{0\} \mid T \in A\}, \\
S_2 &= \{B_i \mid 0 \leq i < \lceil \log_2 n \rceil\}, \\
B_i &= \{v_j \mid \text{the } i\text{th bit of } j \text{ is equal to } 1\} \times \{0, 1\}.
\end{aligned}$$

Hint 2: You might have to use duplication in order to turn the above into a log-approximation-preserving reduction.

Solution:



We modify the reduction in the hint as follows. For a Set Cover instance $V = \{v_0, \dots, v_{n-1}\}$ and $A \subseteq 2^V$, we define:

$$\begin{aligned} f(V, A) &= (U, S), \\ U &= V \times \left\{0, \dots, \lceil \log_2 n \rceil\right\}, \\ S &= R \cup \bigcup_{0 \le k < \lceil \log_2 n \rceil} S_k, \\ S_k &= \{T \times \{k\} \mid T \in A\}, \\ R &= \{B_i \mid 0 \le i < \lceil \log_2 n \rceil\}, \\ B_i &= \{v_j \mid \text{the } i\text{th bit of } j \text{ is equal to } 1\} \times \left\{0, \dots, \lceil \log_2 n \rceil\right\}. \end{aligned}$$

Fix a discerning collection $S' \subset S$. Then S' can be written in the form

$$S' \subset R \cup \bigcup_{0 \le k < \lceil \log_2 n \rceil} S'_k,$$

where S'_k is a set cover of $V \times \{k\}$. Let k^* be the index minimizing $|S'_{k^*}|$, and define $g(S') \subset A$ to be the projection of S'_{k^*} onto V.

So g maps solutions S' of the Questions instance $f(V, A)$ to solutions $g(S') \subset A$ to the original Set Cover instance (V, A). Clearly both f and g are computable in polynomial time. It remains to verify that they preserve log-approximations. We can compute the quantities:

$$\begin{aligned} |f(V, A)| &= \lceil \log_2 n \rceil (|V| + |A| + 2), \\ |S'| &\ge \lceil \log_2 n \rceil (|g(S')| + 1), \\ OPT(f(V, A)) &= \lceil \log_2 n \rceil (OPT(V, A) + 1). \end{aligned}$$

Let $\alpha \in O(\log)$; we can assume α is monotonically increasing. Suppose that S' is an α-approximation for $f(V, A)$:

$$\begin{aligned}
\frac{|S'|}{OPT(f(V,A))} &\leq \alpha(|f(x)|), \\
\frac{\lceil \log_2 n \rceil (|g(S')| + 1)}{\lceil \log_2 n \rceil (OPT(V,A) + 1)} &\leq \alpha\left(\lceil \log_2 n \rceil (|V| + |A| + 2)\right), \\
\frac{|g(S')|}{OPT(V,A) + 1} &\leq \alpha\left(\lceil \log_2 n \rceil (|V| + |A| + 2)\right), \\
\frac{|g(S')|}{OPT(V,A)} &\leq \alpha\left(\lceil \log_2 n \rceil (|V| + |A| + 2)\right) \left(1 + \frac{1}{OPT(V,A)}\right) \\
&\leq \beta(|V| + |A|).
\end{aligned}$$

So we find that $g(S')$ is a β-approximation for (V, A) where $\beta(z) = 2\alpha(z(z+2))$. The proof is complete since $\beta \in O(\log)$.