Problem Set 10 (Optional)

Due: Monday, November 20, 2023 at noon

Problem 10.1 [Twenty Questions].

Let U be a finite set ("the universe"). For two distinct elements $x, y \in U$, a set $T \subset U$ ***separates*** x from y if either:

- $x \in T$ and $y \notin T$; or
- $x \notin T$ and $y \in T$.

A collection $S' \subseteq 2^U$ of subsets of U is ***discerning*** if, for every two distinct elements $x, y \in U$, there exists some $T \in S'$ that separates x from y.

The Minimum Questions problem is, given a set U and a collection $S \subseteq 2^U$ of subsets of U, to find the smallest possible discerning collection $S' \subset S$; that is, the discerning collection consisting of the smallest possible number of subsets.

Prove that there is a ***log-approximation-preserving*** reduction f, g from Minimum Set Cover to Minimum Questions, in the following sense:

- f is a function from Set Cover instances to Questions instances.
- g is a function from solutions of $f(x)$ to solutions of x.
- f, g are computable in polynomial-time.
- For all functions $\alpha \in O(\log n)$ there exists a function $\beta \in O(\log n)$ satisfying the following. For all instances x and all solutions y' of $f(x)$, if y' is an α-approximation for $f(x)$ i.e. $\frac{\mathrm{cost}(y')}{OPT(f(x))} \leq \alpha(|f(x)|)$, then $g(y')$ is a β-approximation for x i.e. $\frac{\mathrm{cost}(g(y'))}{OPT(x)} \leq \beta(|x|)$.

Such a reduction implies that, if you have an $O(\log n)$-approximation algorithm for Minimum Questions, then you can obtain an $O(\log n)$-approximation algorithm for Minimum Set Cover, which is known to be impossible unless P = NP.

You must include a drawing or diagram in your submission.

Hint 1: There is a reduction from Set Cover to the decision version Questions as follows. Suppose the Set Cover instance has elements $V = \{v_0, \dots, v_{n-1}\}$ and sets $A \subseteq 2^V$. Then the reduction to Questions is defined by:

$$\begin{aligned}
U &= V \times \{0, 1\}, \\
S &= S_1 \cup S_2, \\
S_1 &= \{T \times \{0\} \mid T \in A\}, \\
S_2 &= \{B_i \mid 0 \leq i < \lceil \log_2 n \rceil\}, \\
B_i &= \{v_j \mid \text{the } i\text{th bit of } j \text{ is equal to } 1\} \times \{0, 1\}.
\end{aligned}$$

Hint 2: You might have to use duplication in order to turn the above into a log-approximation-preserving reduction.