6.890	Lecture 22	Nov. 20, 2014

(guest lecture by Costis Daskalakis)

PPAD: definition later – start with motivation

Motivation 1: Economic Game Theory

Game:

- n players $1, 2, \ldots, n$
- for each player p: set S_p of strategies
- payoff for each player p:

$$u_p : S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}$$

- e.g. Penalty Shot Game

Nash Equilibrium = locally optimal product distribution of strategies for the players such that no one player can (by changing just their strategy) improve their expected payoff

i.e. $x_1, x_2, \ldots, x_n$ such that $\forall p$:

$$E[u_p(x_1, \ldots, x_p, \ldots, x_n)] \geq E[u_p(x_1, \ldots, x_p', \ldots, x_n)] \quad \forall x_p' \in D(S_p)$$

- e.g. ½ – ½ strategies in Penalty Shot Game
- exist in 2-player zero-sum games [von Neumann 1928]
  - via linear programming
- exist in n-player games [Nash 1950]
  - still no poly-time algorithm to find them

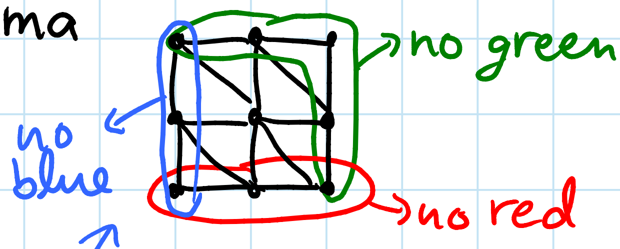
Motivation 2: Brouwer's Fixed-Point Theorem

for any convex, closed, bounded set S, any continuous map $f: S \to S$ has a fixed point $p \in S$: $f(p) = p$ [Brouwer 1910]

Nash's proof via Brouwer's Theorem

- $f: [0,1]^n \to [0,1]^n$ is essentially a vector field inticating how each player can improve their mixed strategy (distribution)
- fixed point of f = Nash equilibrium

Motivation 3: Sperner's Lemma



- square grid graph + backslash diagonals
- assign vertices 3 colors

2D version: if boundary is legally colored then there are an odd number ($\Rightarrow \geq 1$) of trichromatic Δ

d-dimensional version too (not covered here)

Proof of Brouwer via Sperner:

- for all ε, show approximate fixed point: $|f(x)-x| < \varepsilon$ via Sperner's Lemma
  - color points according to direction of $f(x)-x$ (which of 3 boundaries)
- use compactness to take limit $\varepsilon \to 0$
  (may not preserve oddness of solution count)

Computational version of Sperner:

$x, y \to \boxed{C} \to R/G/B$

- grid of size $2^n \times 2^n$
- internal vertex colors given by circuit C
- boundary in canonical legal coloring
- goal: find trichromatic Δ

Computational version of Nash:

- given # players n, enumeration of strategy set S_p & utility function $u_p : S \to \mathbb{R}$ of every player p.
- goal: ε-Nash equilibrium
  ↳ expected payoff can't improve by more than $+\varepsilon$
  - avoids representation issue for irrational equilibria (required for e.g. $n=3$ game)

as in L15

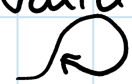
Search problem defined by relation $R \subseteq \{0,1\}^* \times \{0,1\}^*$
where $(x,y) \in R$ means y is solution to x

Total if $\forall x\ \exists y : (x,y) \in R$ i.e. always $\exists \geq 1$ solution
- e.g. Sperner & Nash & Brouwer

FNP = {NP search problems}

FNP-complete = $\in$ FNP & $\exists$ one-call (Karp) reduction from every problem $\in$ FNP
- impossible for total problems reducing from nontotal problem e.g. SAT

Complexity theory for total problems: (TFNP)
- identify combinatorial argument for existence proof
- define complexity class
- check tightness via completeness result

Proof of Sperner's Lemma:
- add artificial trichromatic Δ at boundary
- define directed walk from that Δ: keep crossing bichromatic edges with same 2 colors with same orientation (else find trichromatic Δ)
- can't exit square by valid boundary coloring
- can't form a cycle  (uncolorable)
- for odd number theorem: can walk from every other trichromatic Δ to another $\Rightarrow$ even # except for one from boundary

Directed parity argument:

- vertices of graph represent Δs
- all vertices have in & out degrees ≤ 1

$\Rightarrow$ graph = disjoint union of directed paths, cycles, & isolated vertices

- degree-1 vertex = trichromatic Δ
- degree-2 vertex = walkable (2 bichromatic edges with right orientation)
- degree-0 vertex = rest

Nonconstructive step: if there's an unbalanced vertex then there's another   in-deg. $\neq$ out-deg.

End of the Line:

- each vertex v has candidate incoming & outgoing edge $P(v)$ & $N(v)$
  - given as circuit : $V \to V$ → size 2^n
- actual edge $(v,w) \Leftrightarrow$ both ends agree: $N(v) = w \wedge P(w) = v$
- goal: if 0^n is unbalanced, find another unbalanced node   → checkable in $O(n)$ time (4 circuit evaluations)
- $\in$ FNP: certificate = another unbalanced node

PPAD = { search problems $\in$ FNP reducible to End of the Line }   [Papadimitriou 1994]

So: Nash → Brouwer → Sperner → PPAD

In fact: Nash ← Brouwer ← Sperner ← PPAD

i.e. Nash, $\underbrace{\text{Brouwer, Sperner}}_{\text{[Papadimitriou 1994]}}$ are PPAD-complete

[Daskalakis, Goldberg, Papadimitriou 2006]

– even for 2-player Nash [Chen & Deng 2006]

Proof sketch: generic PPAD

→ embed graph in $[0,1]^3$

→ 3D Sperner

→ Arithmetic Circuit SAT

→ Nash

Arithmetic Circuit SAT:

- input: variable nodes $x_1, \dots, x_n$ ← in degree 1
  gate nodes $\to (:=) \to$ $\quad \rightrightarrows (+) \to$ etc. ← in degree $\in \{0, 1, 2\}$
  cycles allowed
  arbitrary out degrees
- goal: assignment of values $\in [0,1]$ to $x_1, \dots, x_n$
  satisfying all gate constraints:
  - $(x) \to (:=) \to (y) \Rightarrow y = x$
  - $(x), (y) \to (+) \to (z) \Rightarrow z = x + y$ $\quad (x), (y) \to (-) \to (z)$ ditto
  - $(c) \to (x) \Rightarrow x = c$ $\quad$ } for constant
  - $(x) \to (\times c) \to (y) \Rightarrow y = c \cdot x$ } $c \in [0,1]$
  - $(x), (y) \to (>) \to (z) \Rightarrow z = \begin{cases} 0 & \text{if } x < y \\ 1 & \text{if } x > y \\ \text{arbitrary} & \text{if } x = y \end{cases}$

    ↖ weird but necessary
- total: always a satisfying assignment ← not obvious
- PPAD-complete

- improvement from exponential (2^{-cn}) noise tolerance
  → polynomial (n^{-c}) noise tolerance [Chen, Deng, Teng 2006]
  "Approximate Arith. Circuit SAT"