

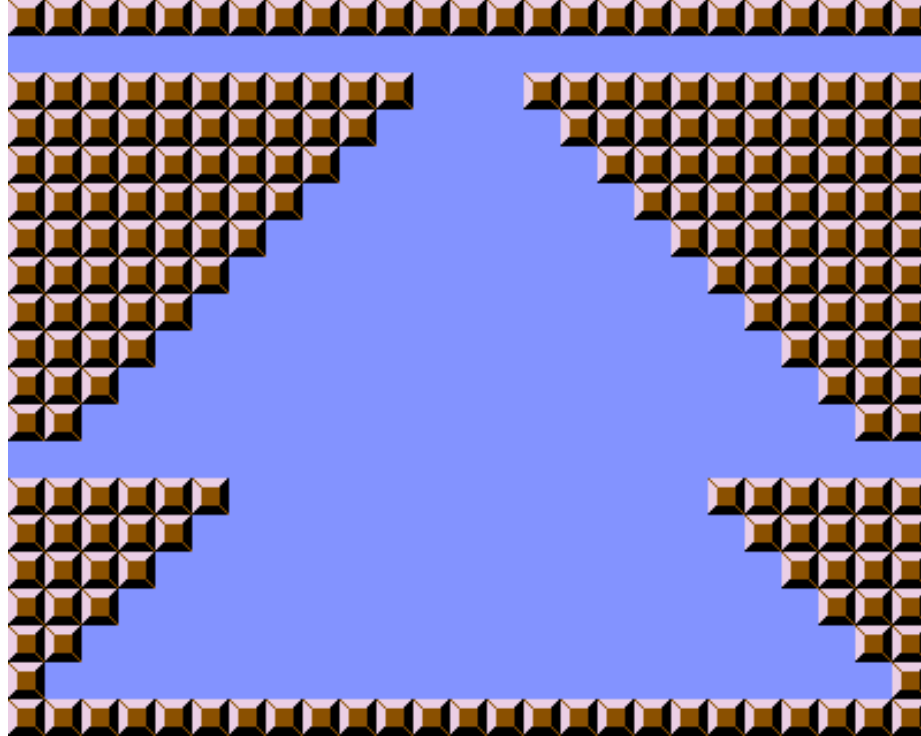


# Super Mario Bros. PSPACE-complete

[Demaine, Viglietta,  
Williams 2016]

crossover

door



traverse ←

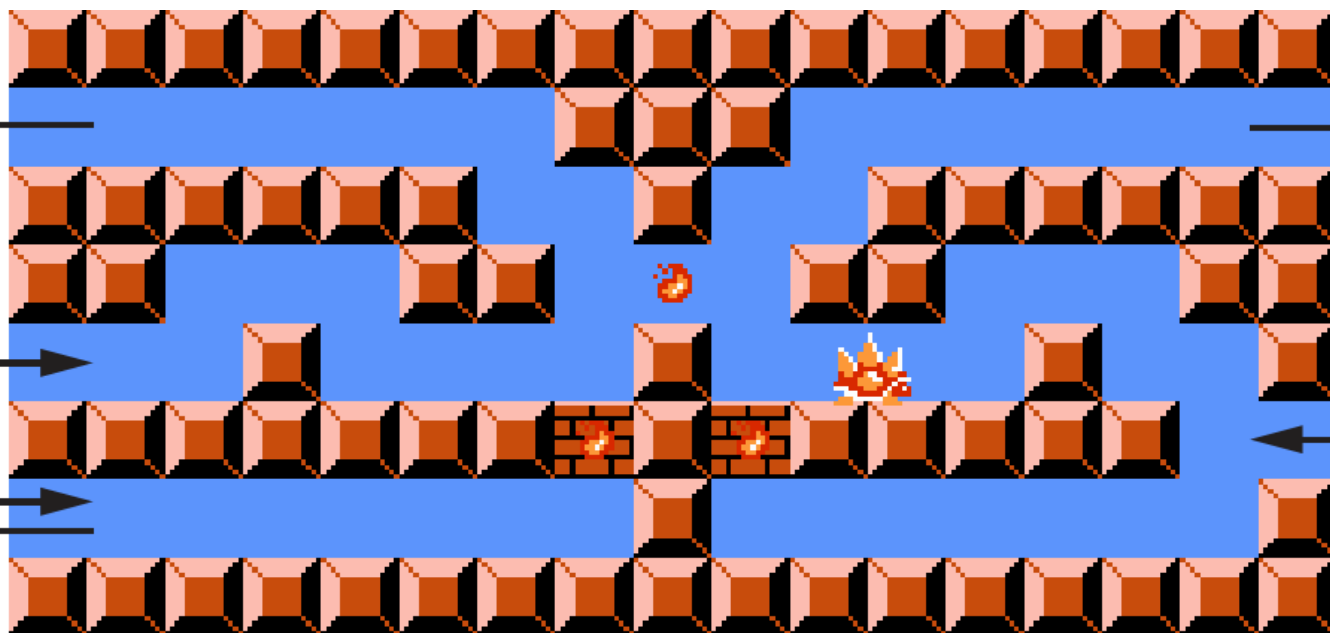
→ close

traverse →

← close

open →

←



MARIO  
000700

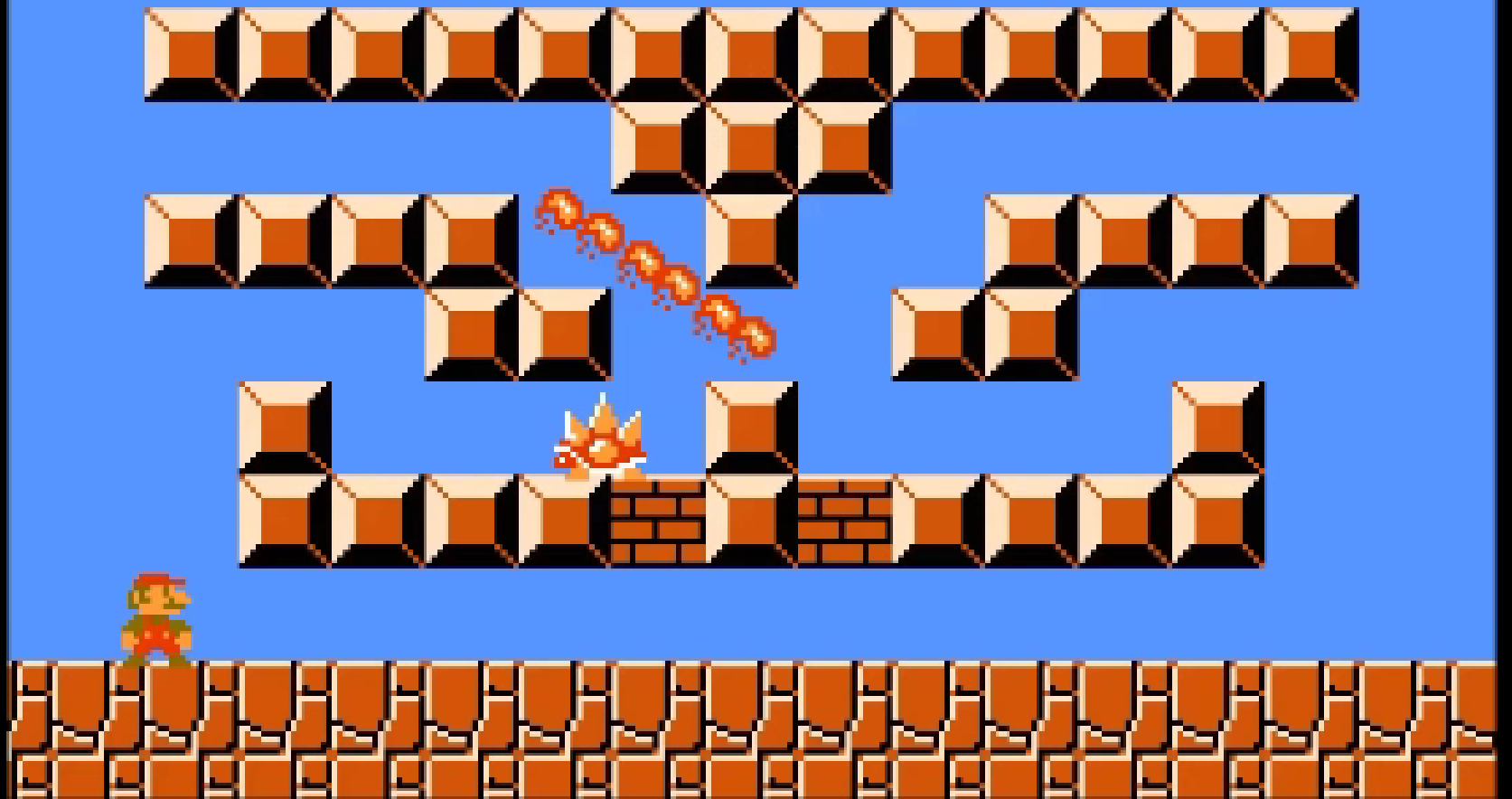
×00



WORLD  
1-1

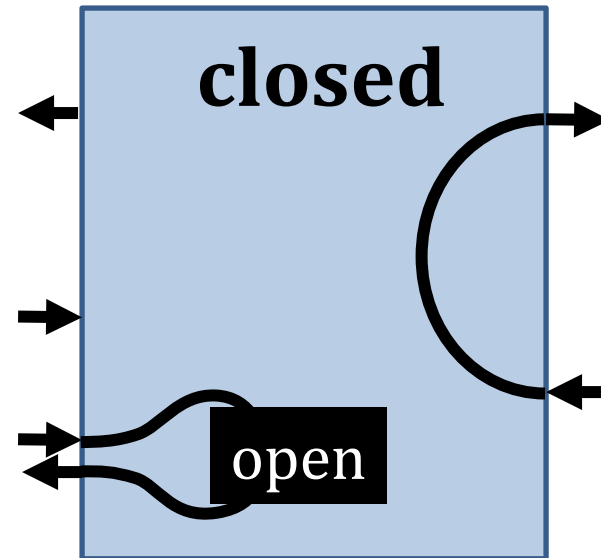
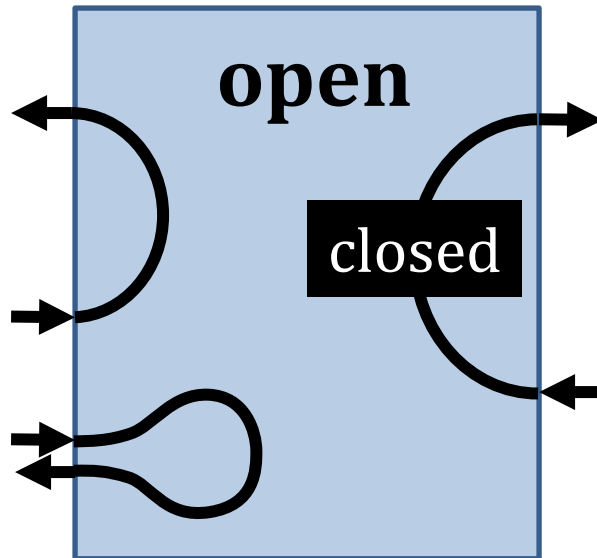
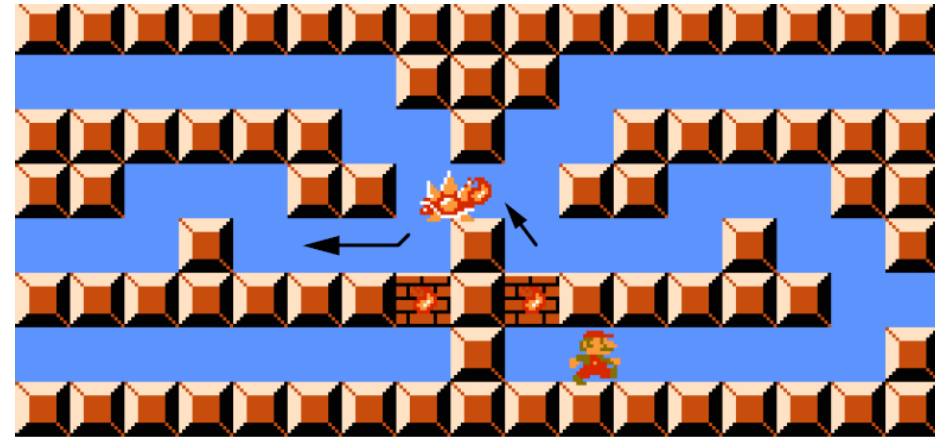
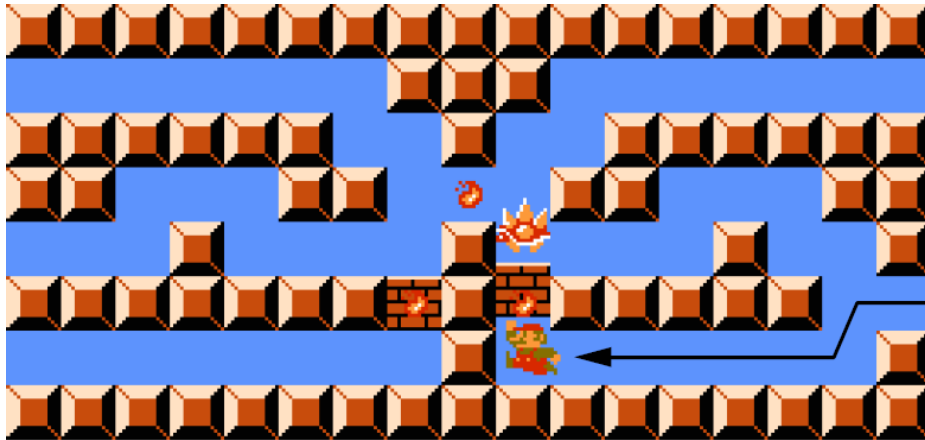
TIME  
381

[Demaine, Viglietta, Williams 2016]



# Mario Door as a Gadget

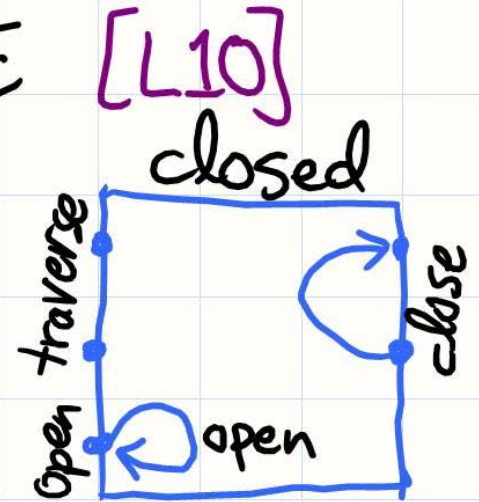
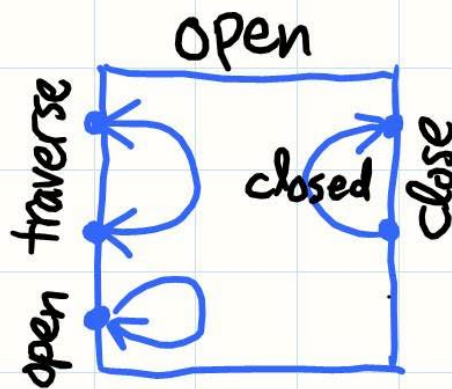
[Demaine, Viglietta, Williams 2016]



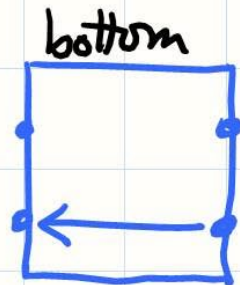
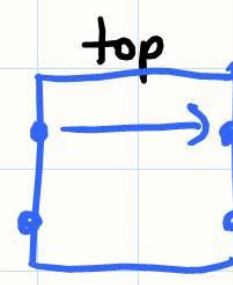
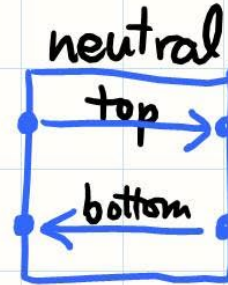
# Example Gadgets

Example: door from Mario PSPACE [L10]

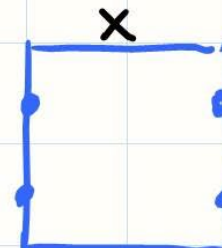
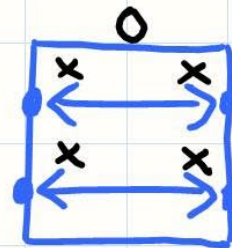
- 2 states
- 5 locations



Example: directed antiparallel NAND [L6]

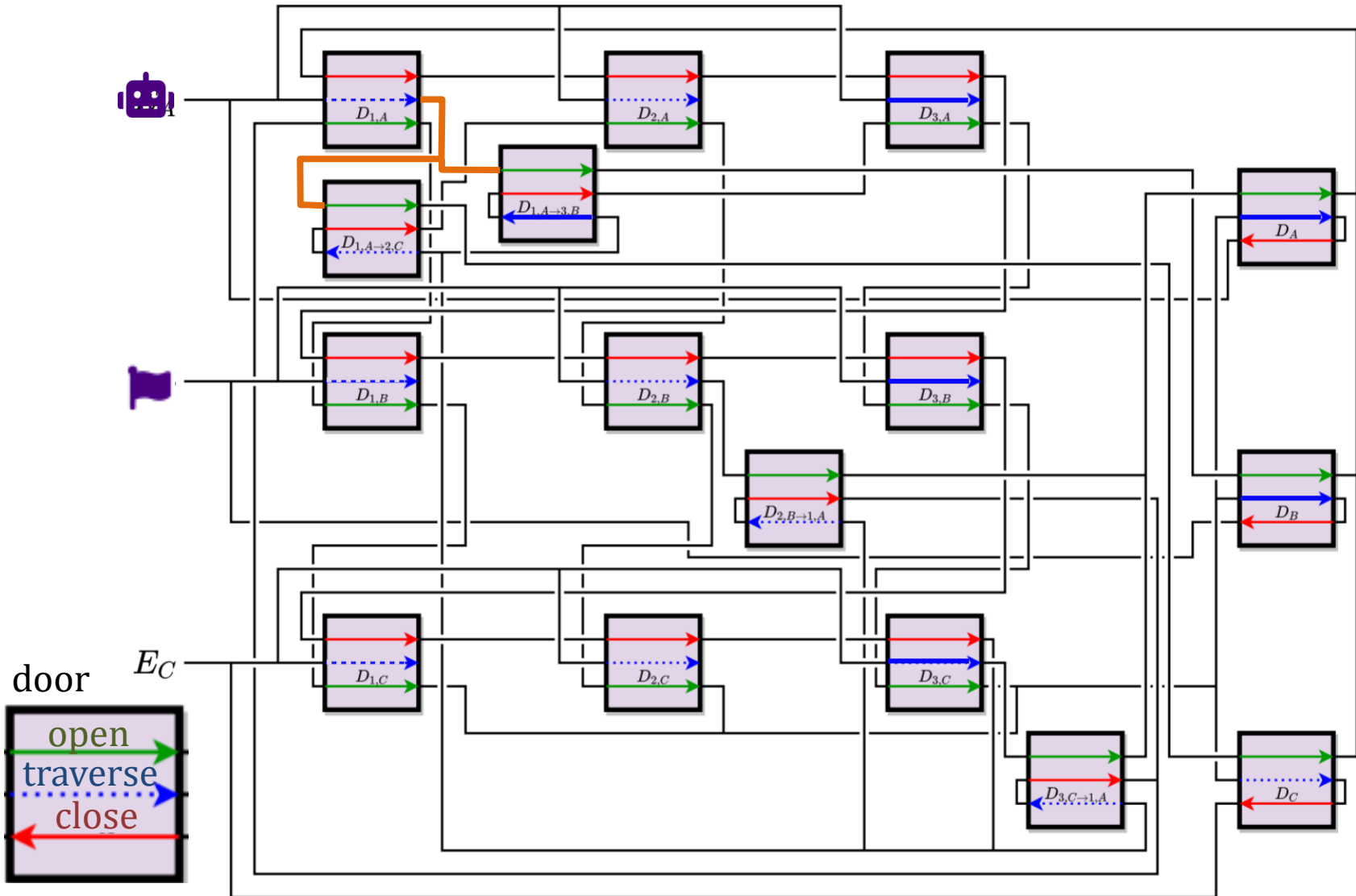


Example: undir. noncrossing matched crumblers [L6]



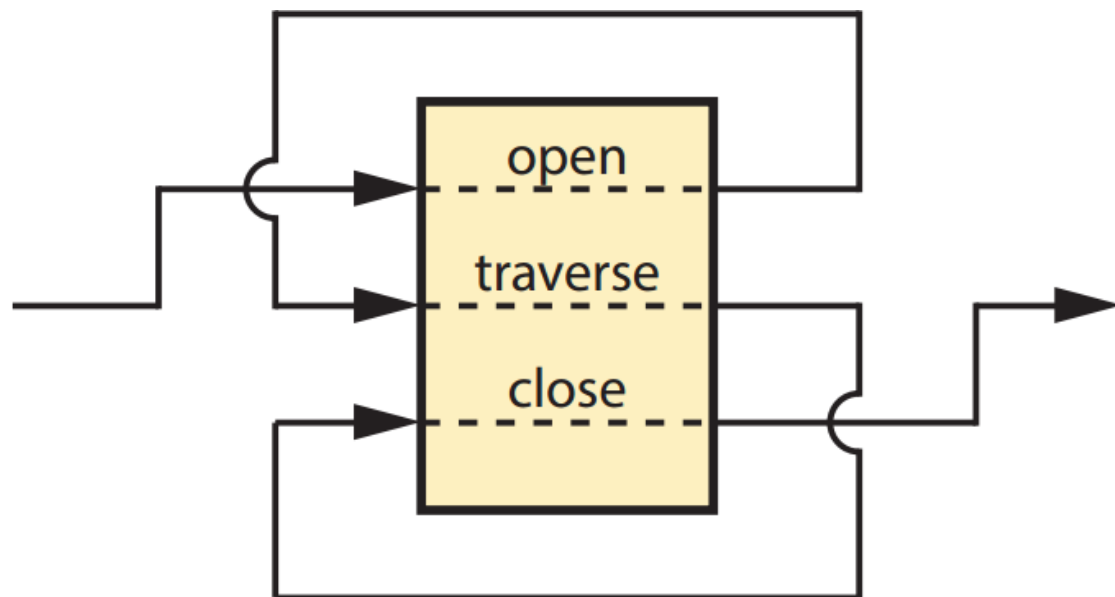
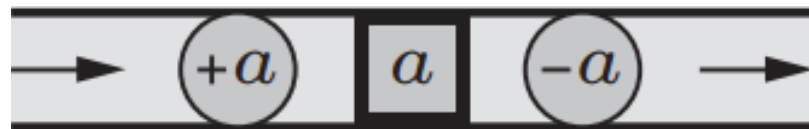
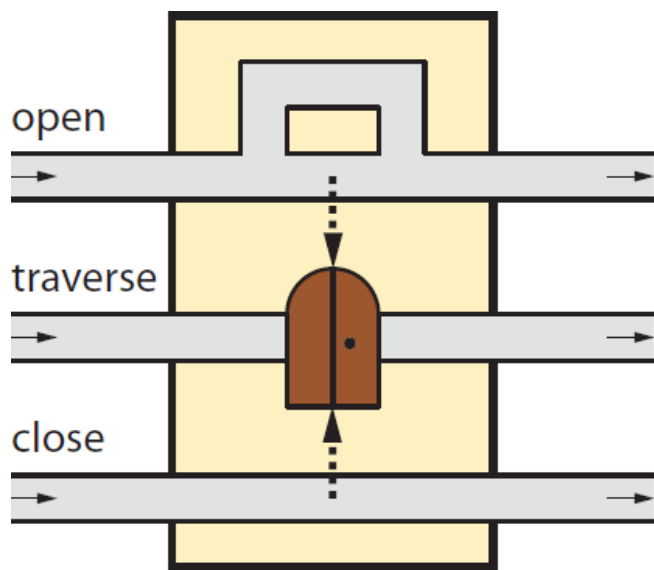
# System of Gadgets

[Demaine,  
Hendrickson,  
Lynch 2020]

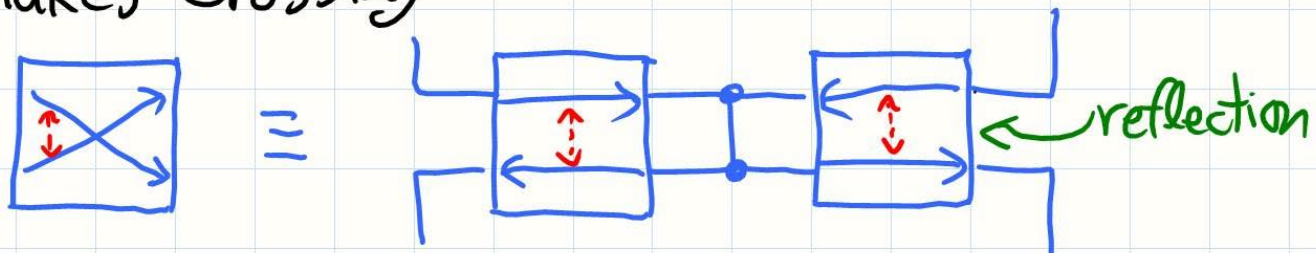


# Doors are PSPACE-complete

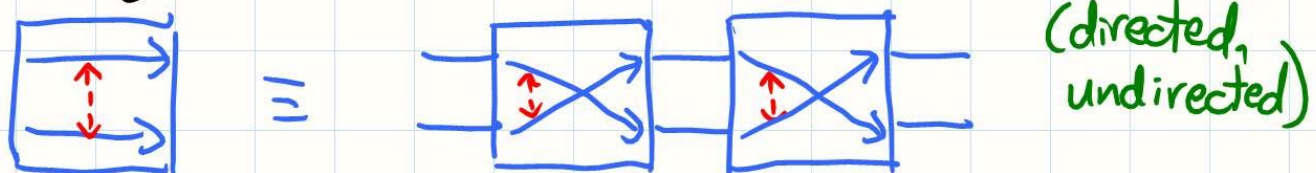
[Aloupis, Demaine, Guo, Viglietta 2014]



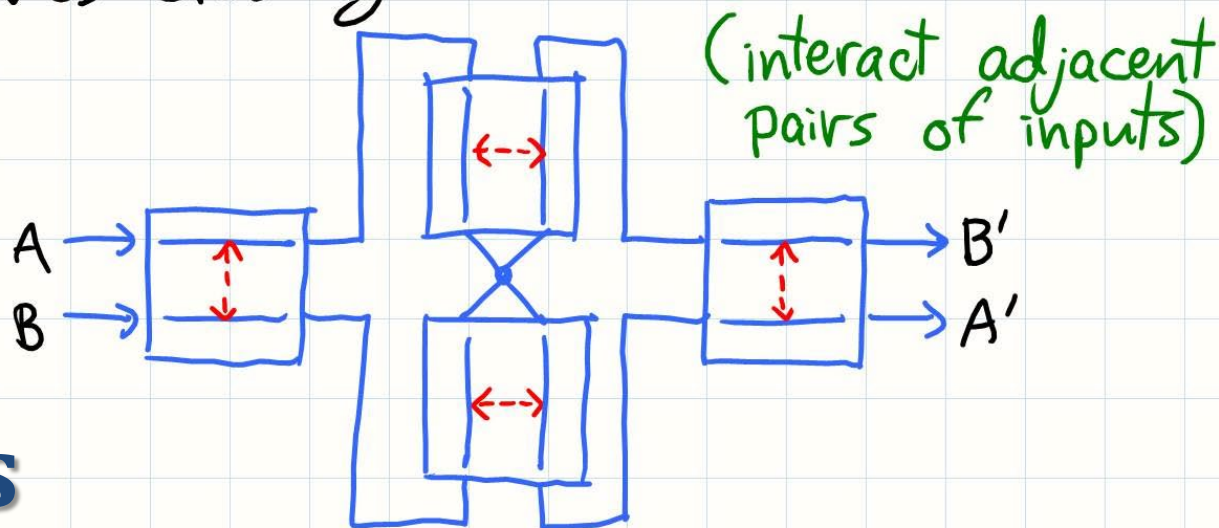
- antiparallel NAND/matched crumblers makes crossing:



- crossing NAND/matched crumblers makes parallel:



- undirected noncrossing NAND matched crumblers makes crossing:

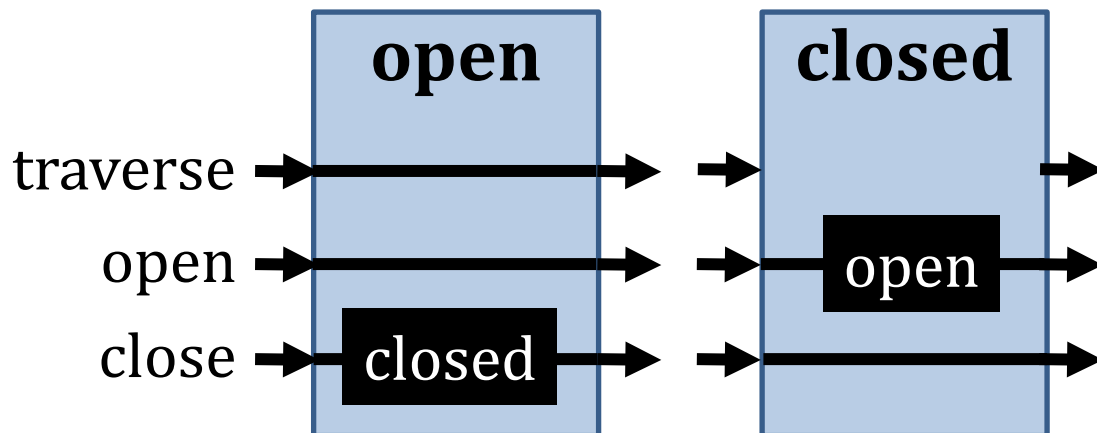


# Gadget Simulations

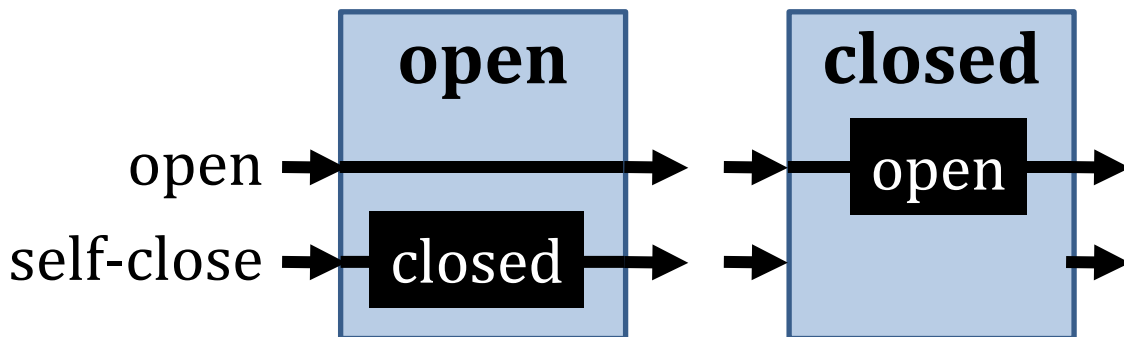


# Door Gadgets

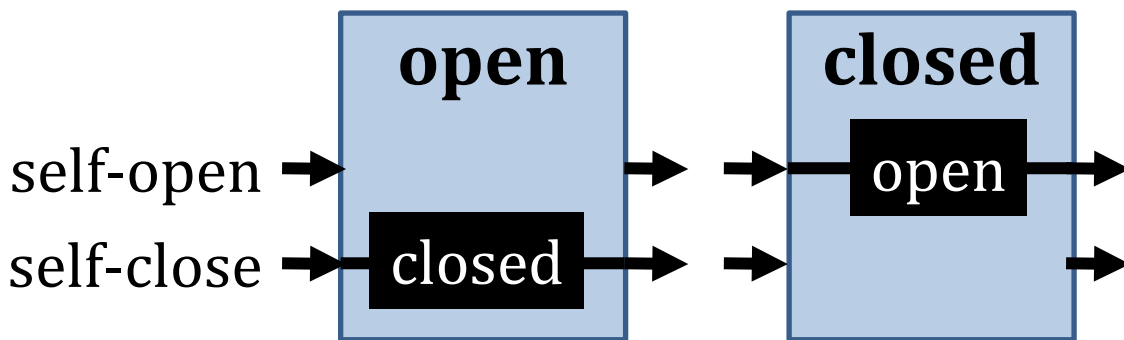
**Door**



**Self-closing door**



**Symmetric self-closing door**

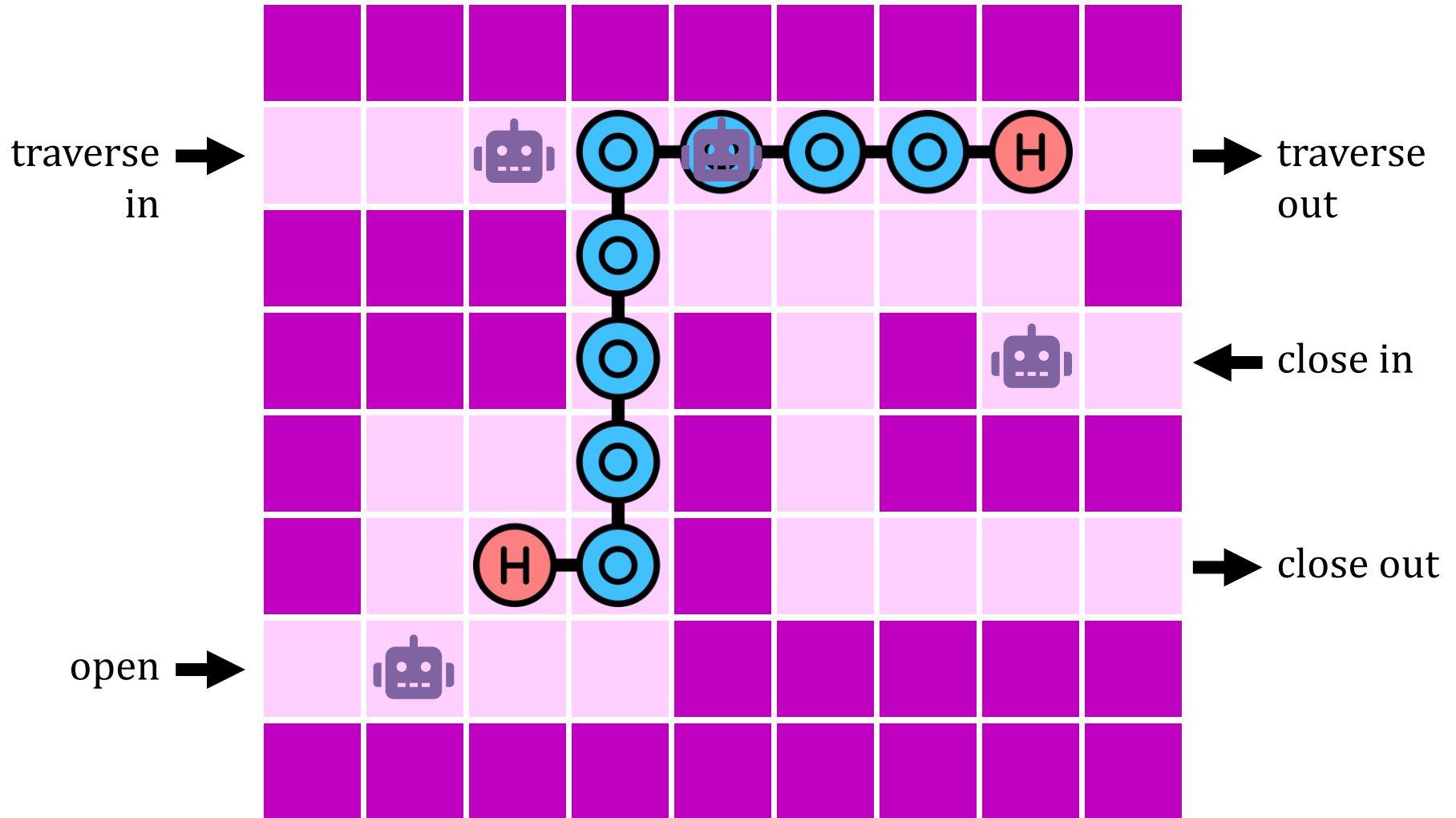






# Sokobond Door

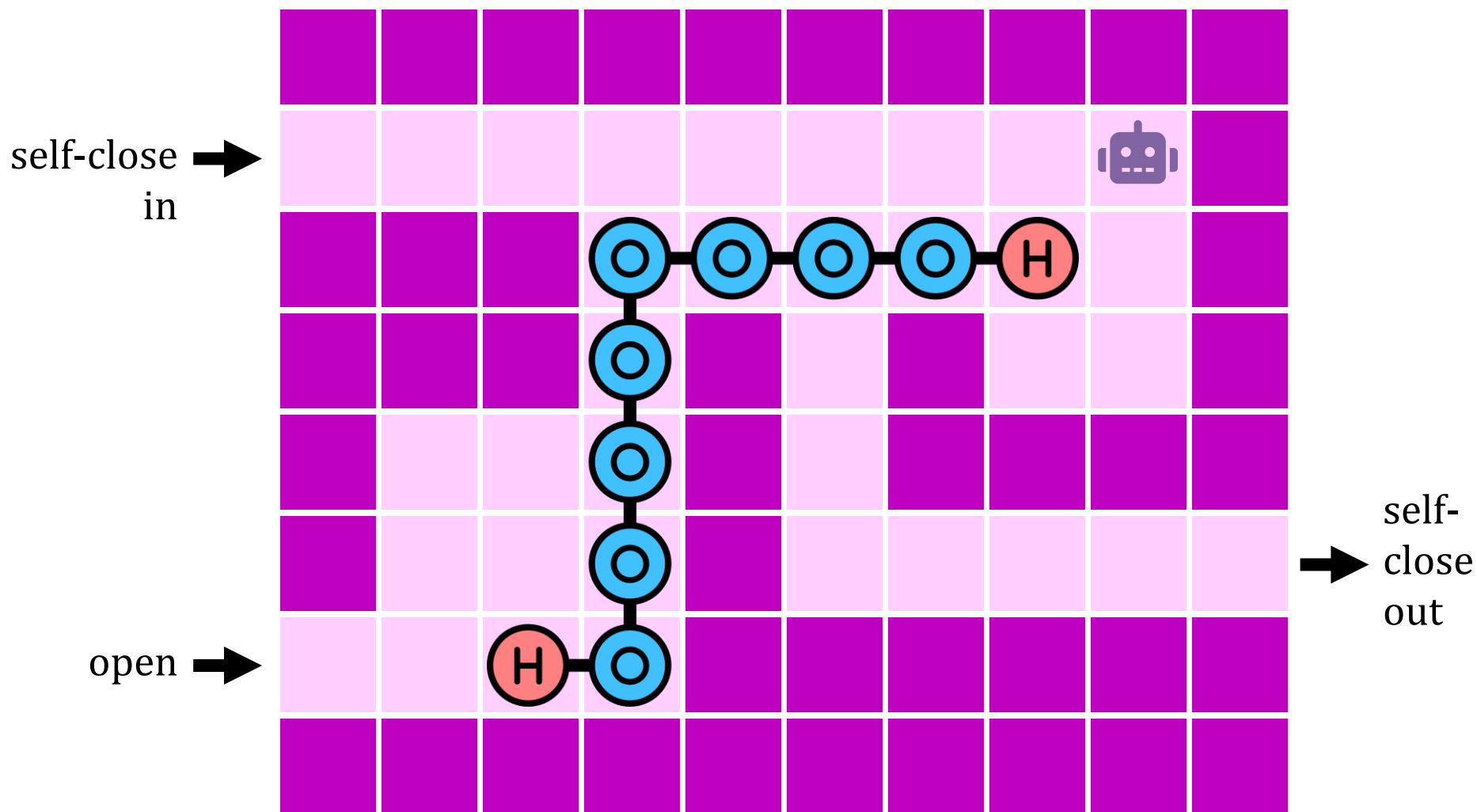
[Ani, Bosboom, Demaine,  
Diomidov, Hendrickson,  
Lynch 2020]





# Sokobond Self-Closing Door

[Ani, Bosboom, Demaine,  
Diomidov, Hendrickson,  
Lynch 2020]



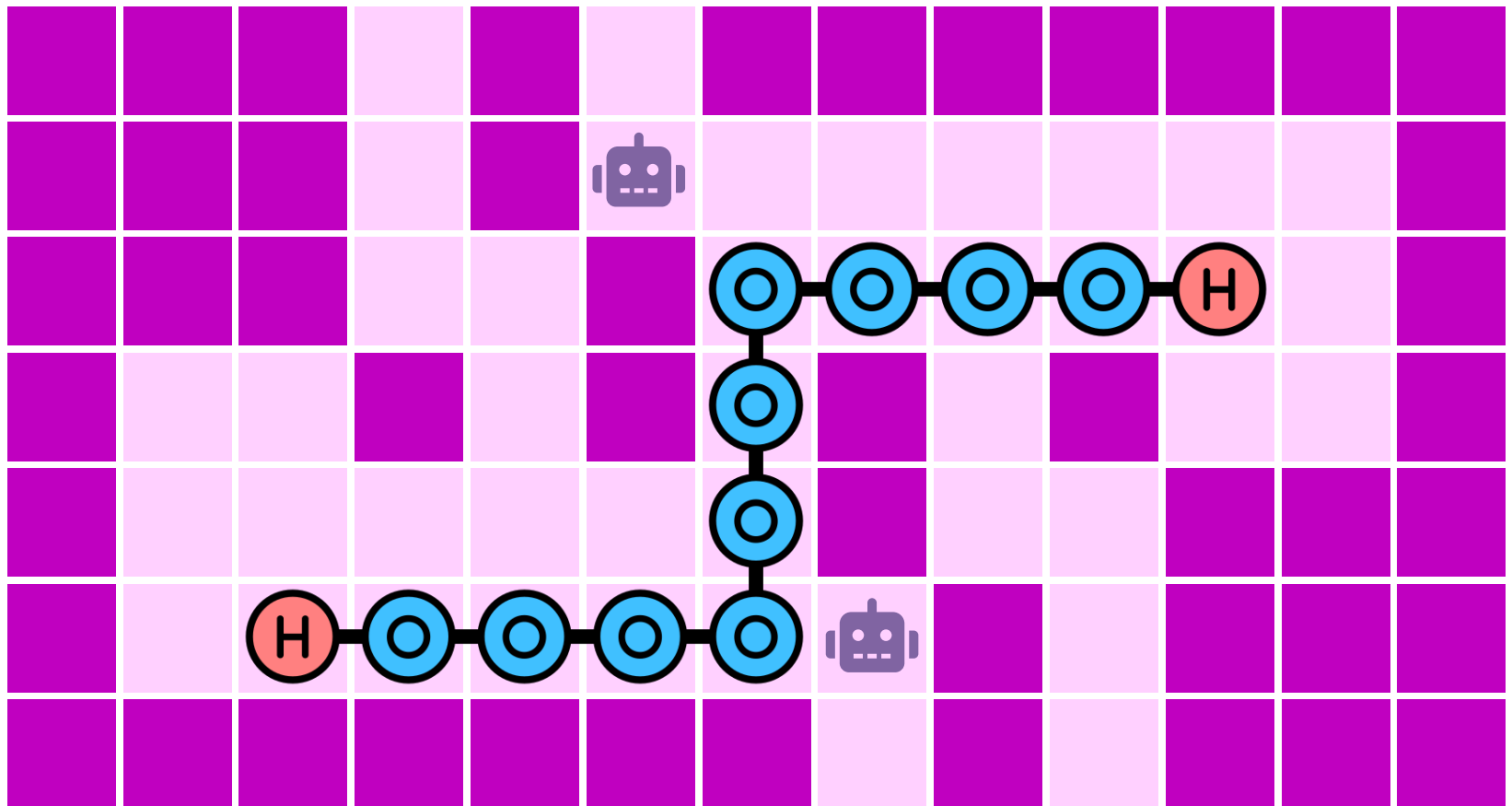


# Sokobond Symmetric Self-Closing Door

[Ani, Bosboom, Demaine,  
Diomidov, Hendrickson,  
Lynch 2020]

self-open out

self-close in



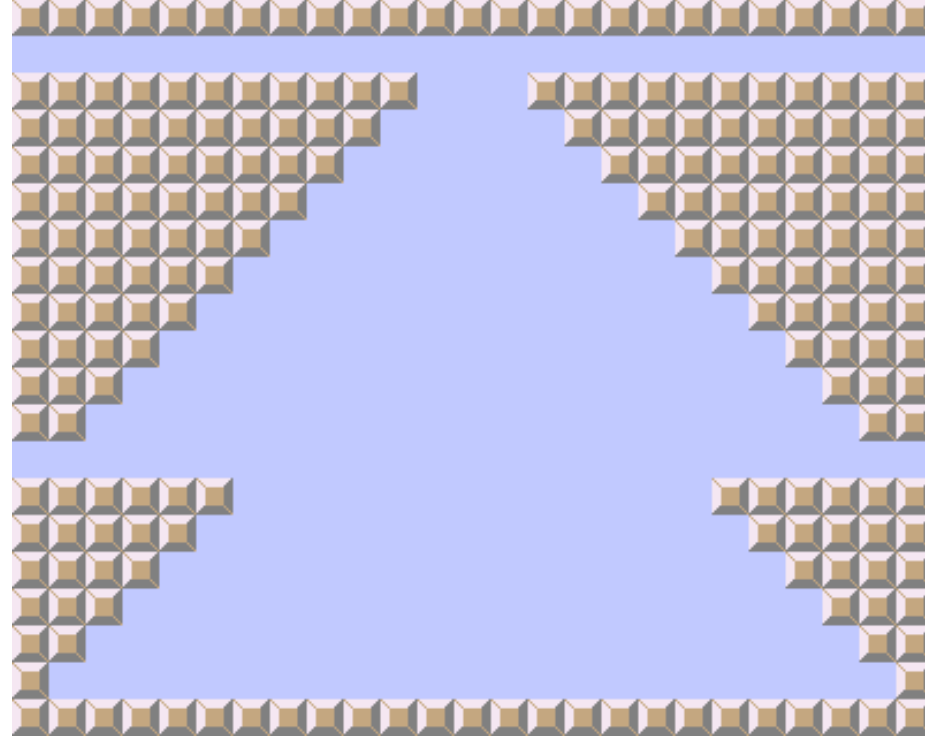
self-open in

self-close out



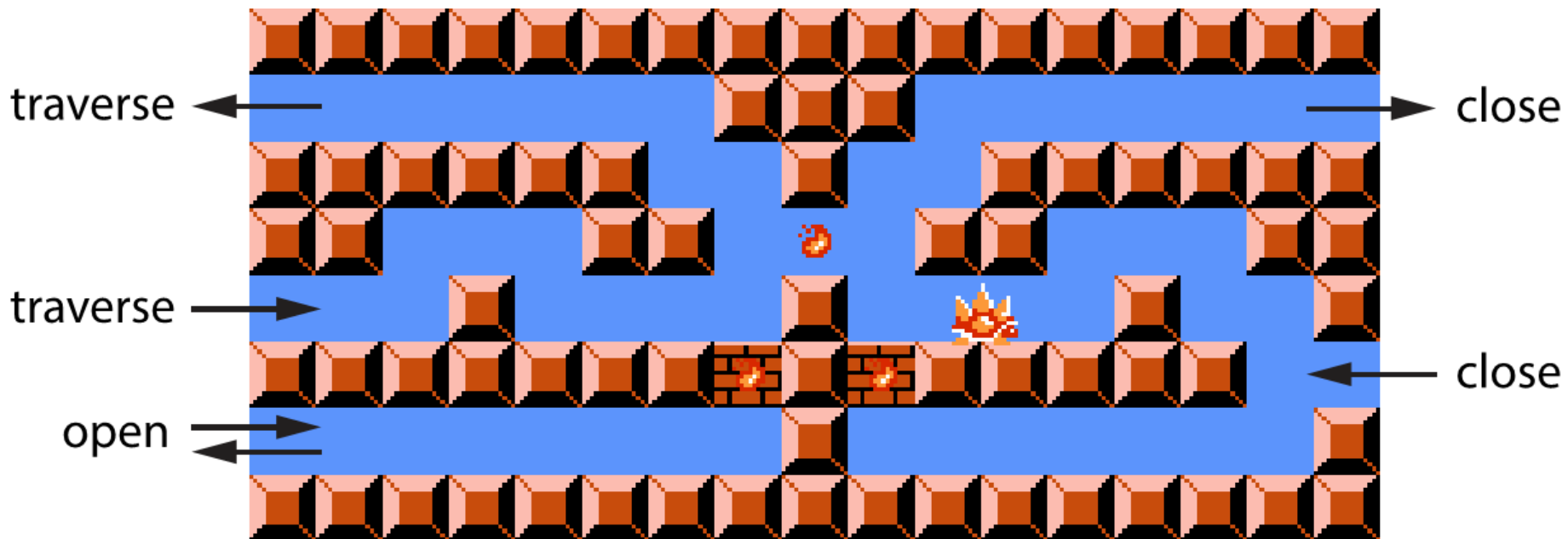
# Super Mario Bros. PSPACE-complete

[Demaine, Viglietta,  
Williams 2016]



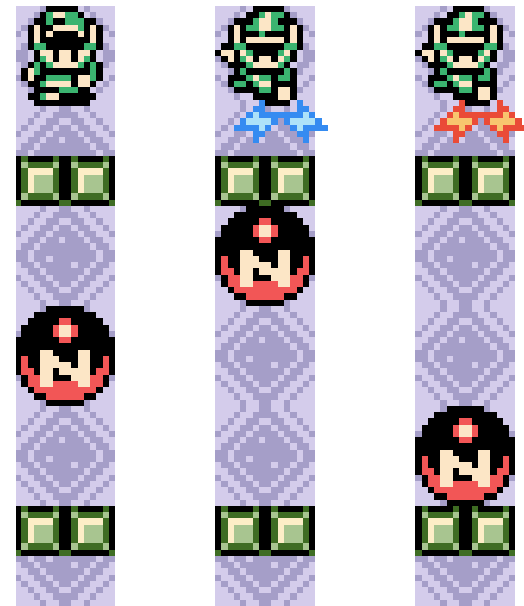
crossover

door

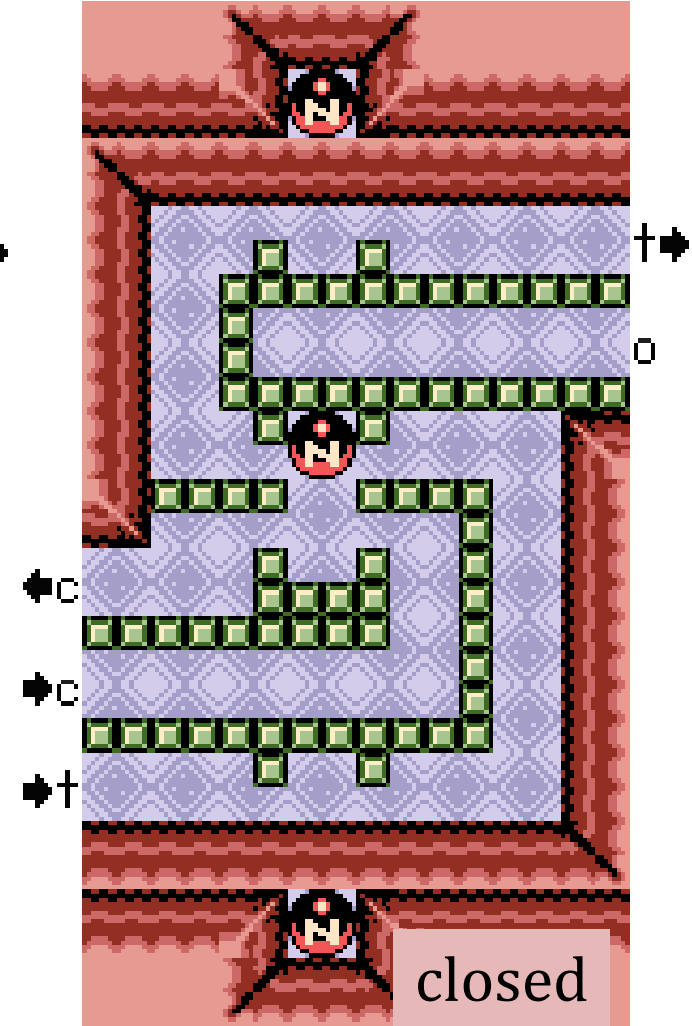
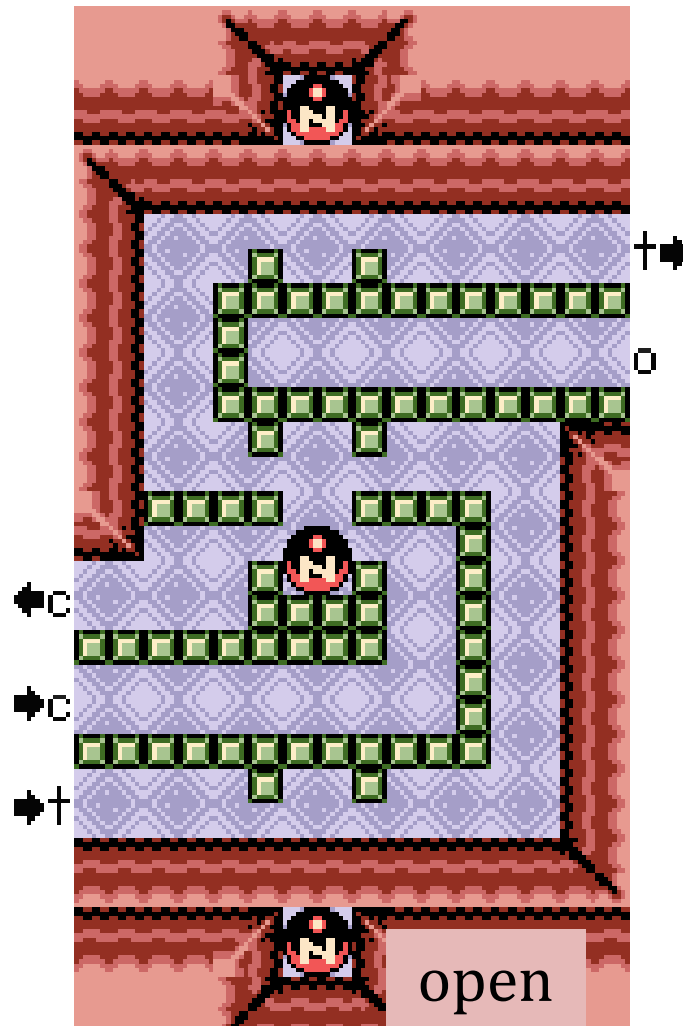


# The Legend of Zelda: Oracle of Seasons is PSPACE-complete

[Bosboom, Brunner, Coulombe, Demaine, Hendrickson, Lynch, Najt 2022]



magnetic gloves

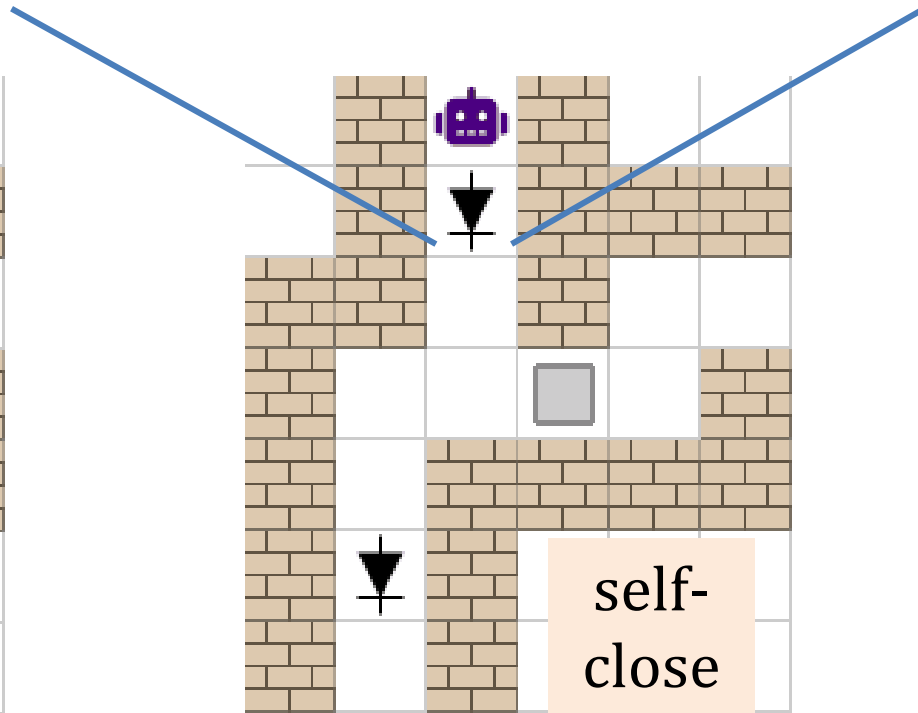
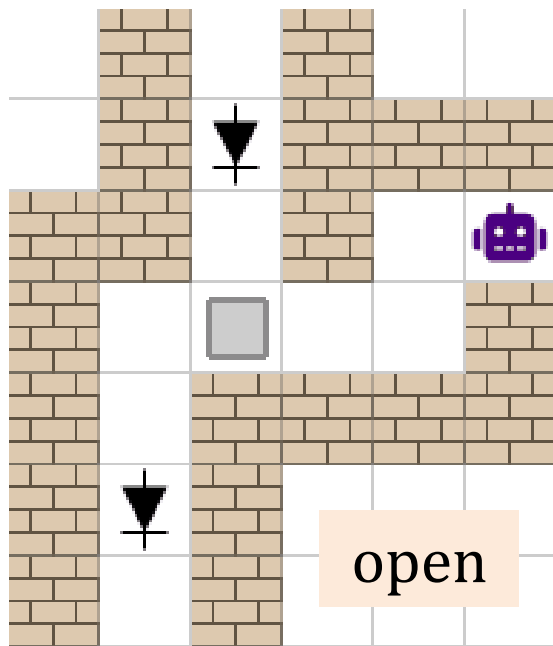
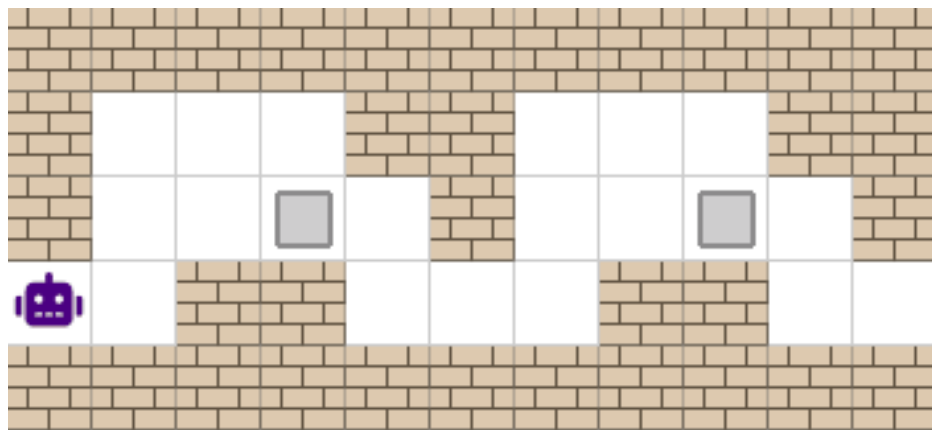




# Pull!-1F(G) is PSPACE-complete

[Ani, Asif, Demaine, Diomidov, Hendrickson, Lynch, Scheffler, Suhl 2020]

self-closing door





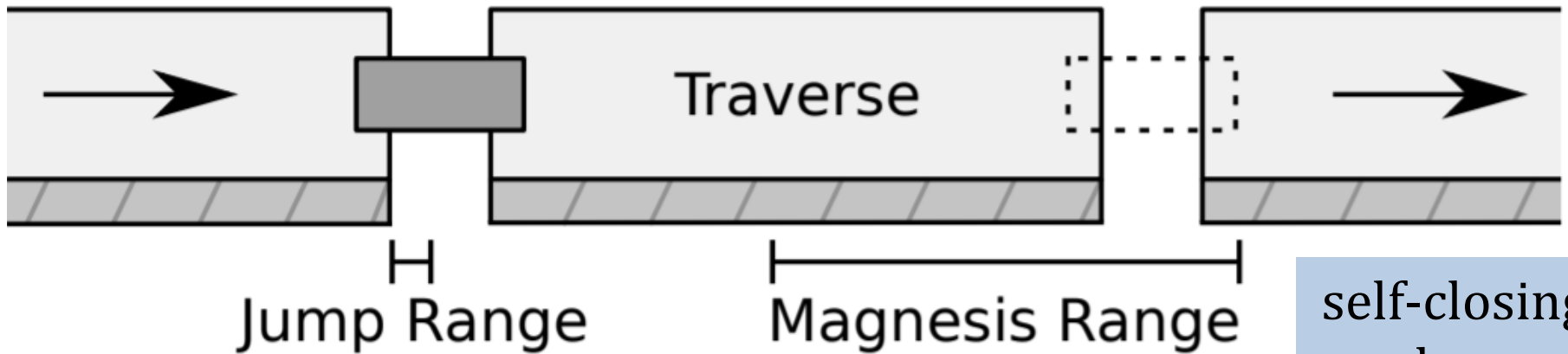
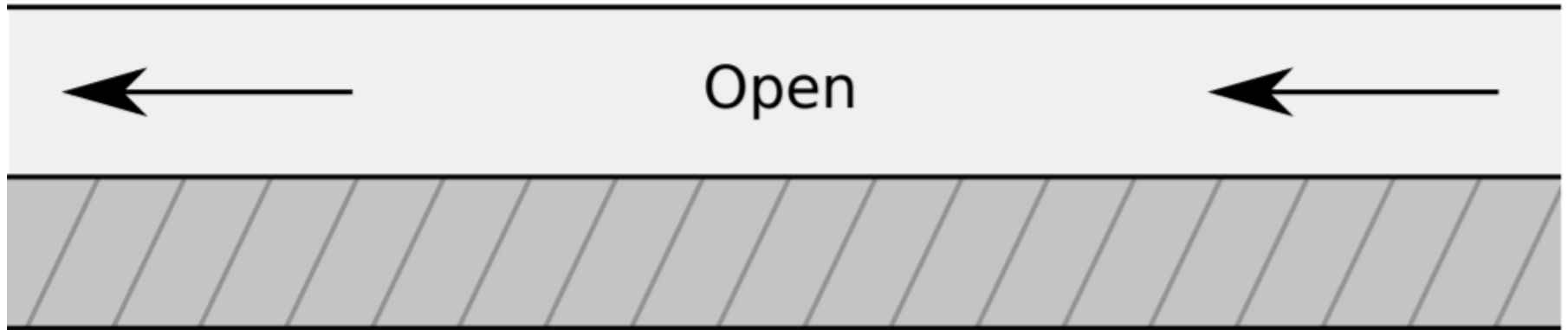
# The Legend of Zelda: Breath of the Wild is PSPACE-complete

[Bosboom, Brunner,  
Coulombe, Demaine,  
Hendrickson, Lynch,  
Najt 2021]



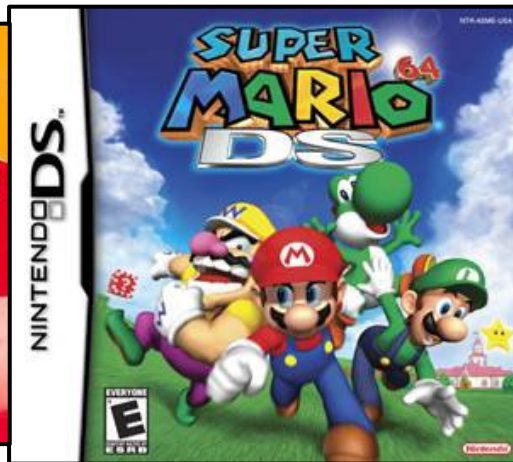
# The Legend of Zelda: Breath of the Wild is PSPACE-complete

[Bosboom, Brunner,  
Coulombe, Demaine,  
Hendrickson, Lynch,  
Najt 2021]

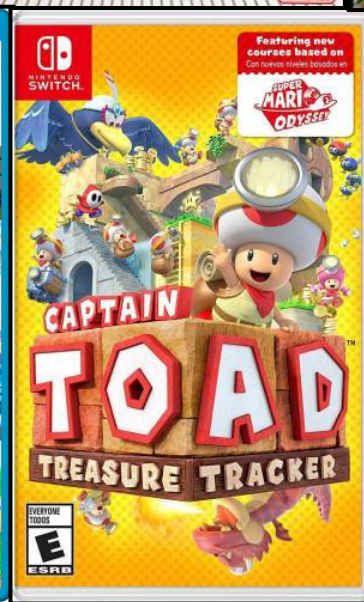
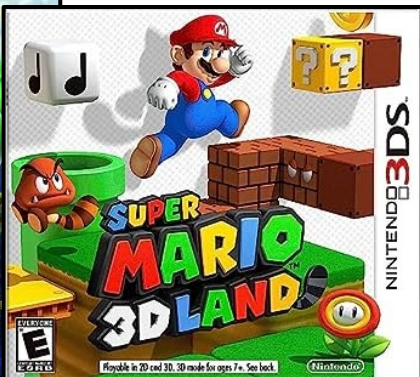




# Applications of Planar Symmetric Self-Closing Door Framework



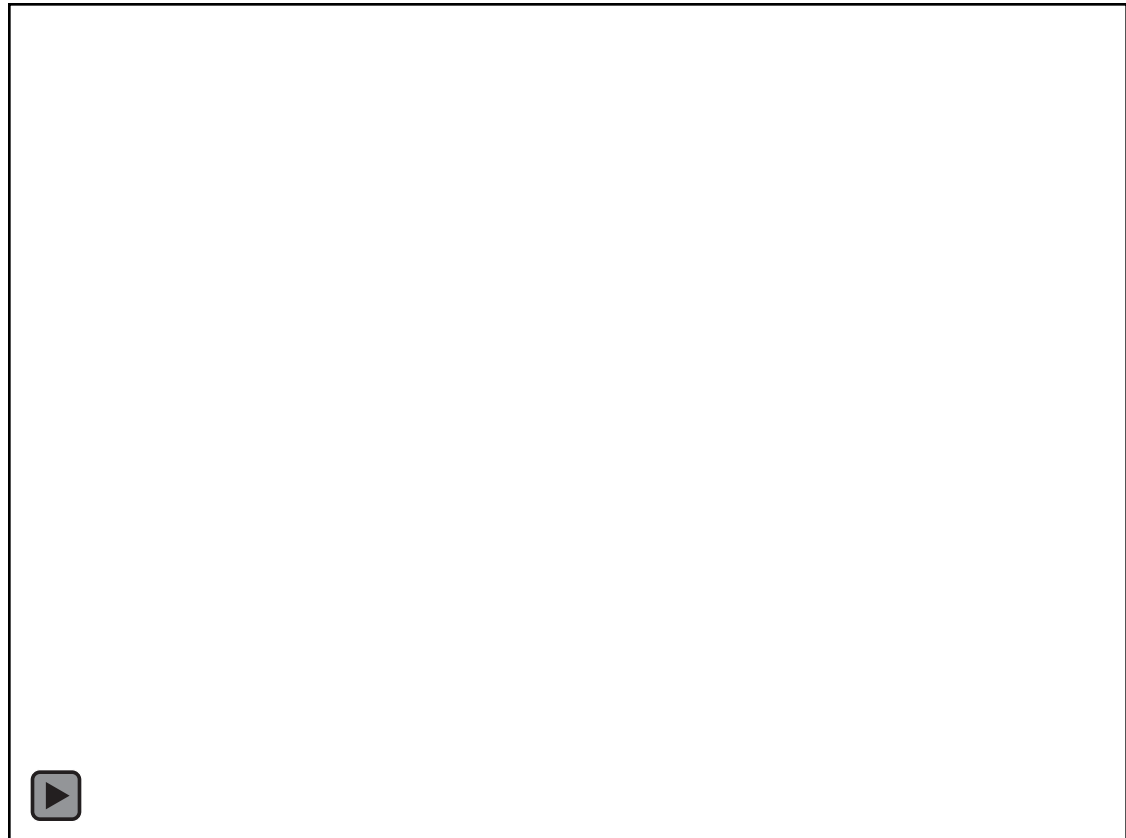
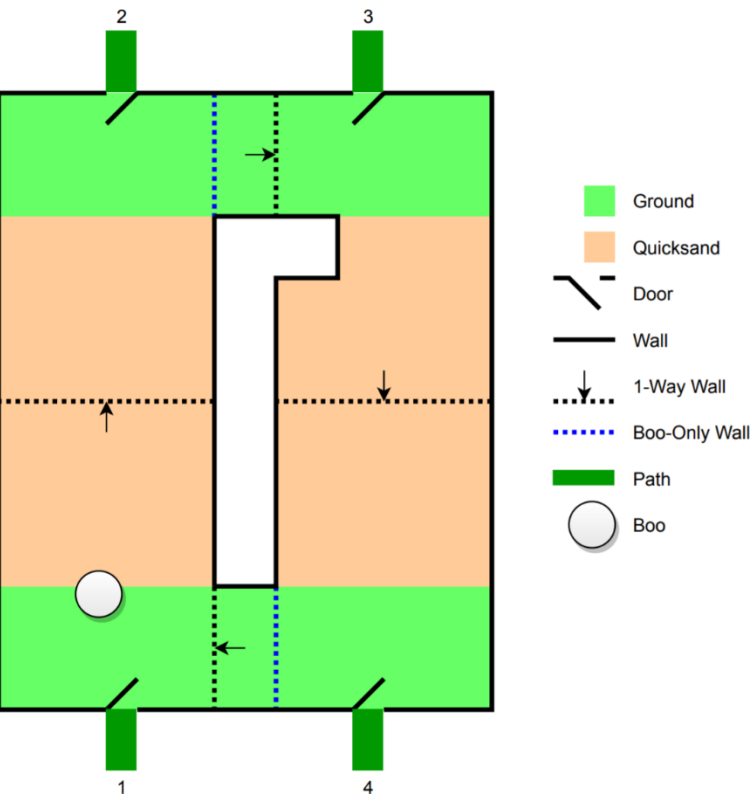
[Ani, Bosboom, Demaine, Diomidov, Hendrickson, Lynch 2020]



# Super Mario 64 is PSPACE-complete

[Ani, Bosboom, Demaine,  
Diomidov, Hendrickson,  
Lynch 2020]

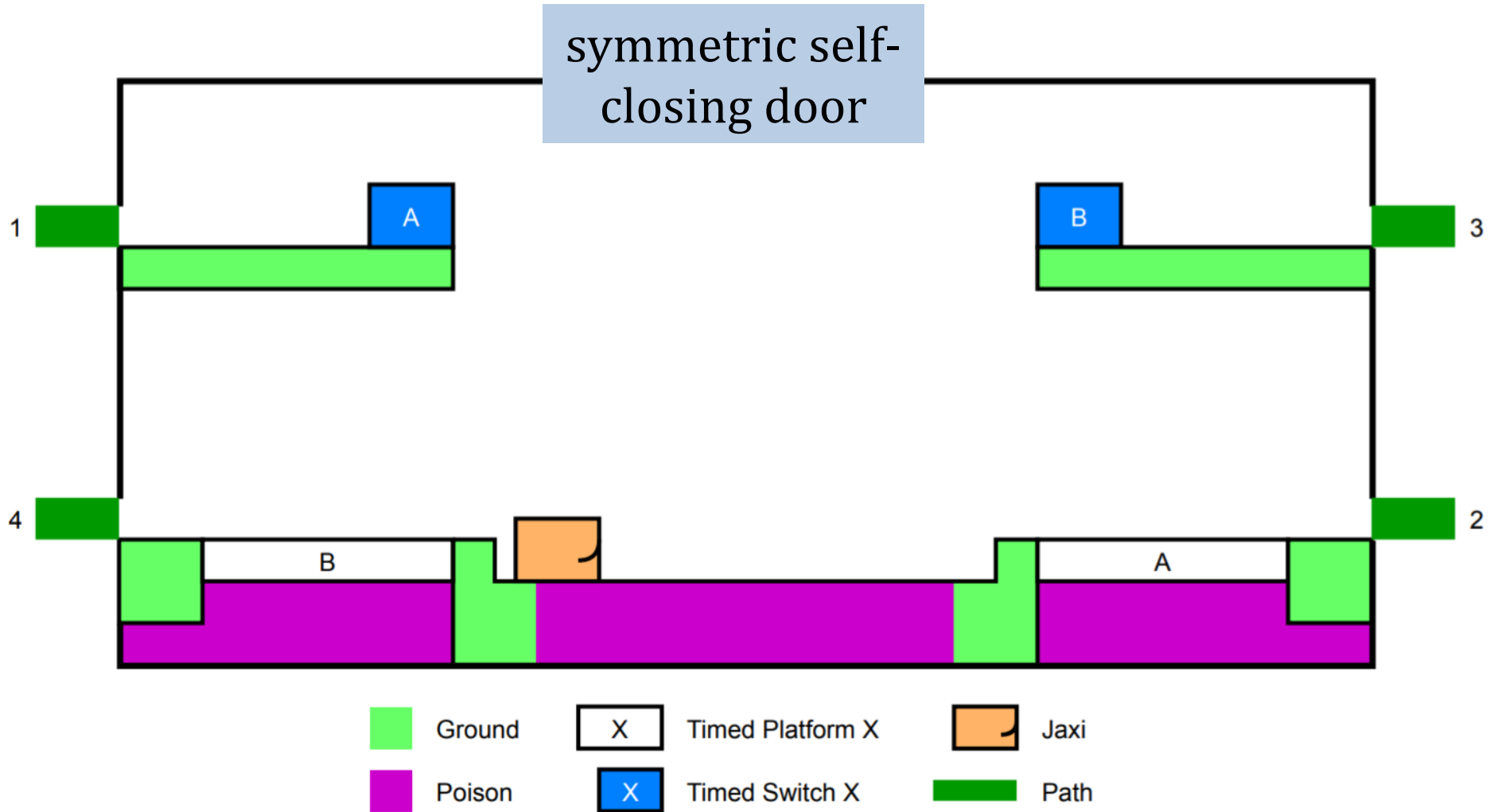
symmetric self-  
closing door





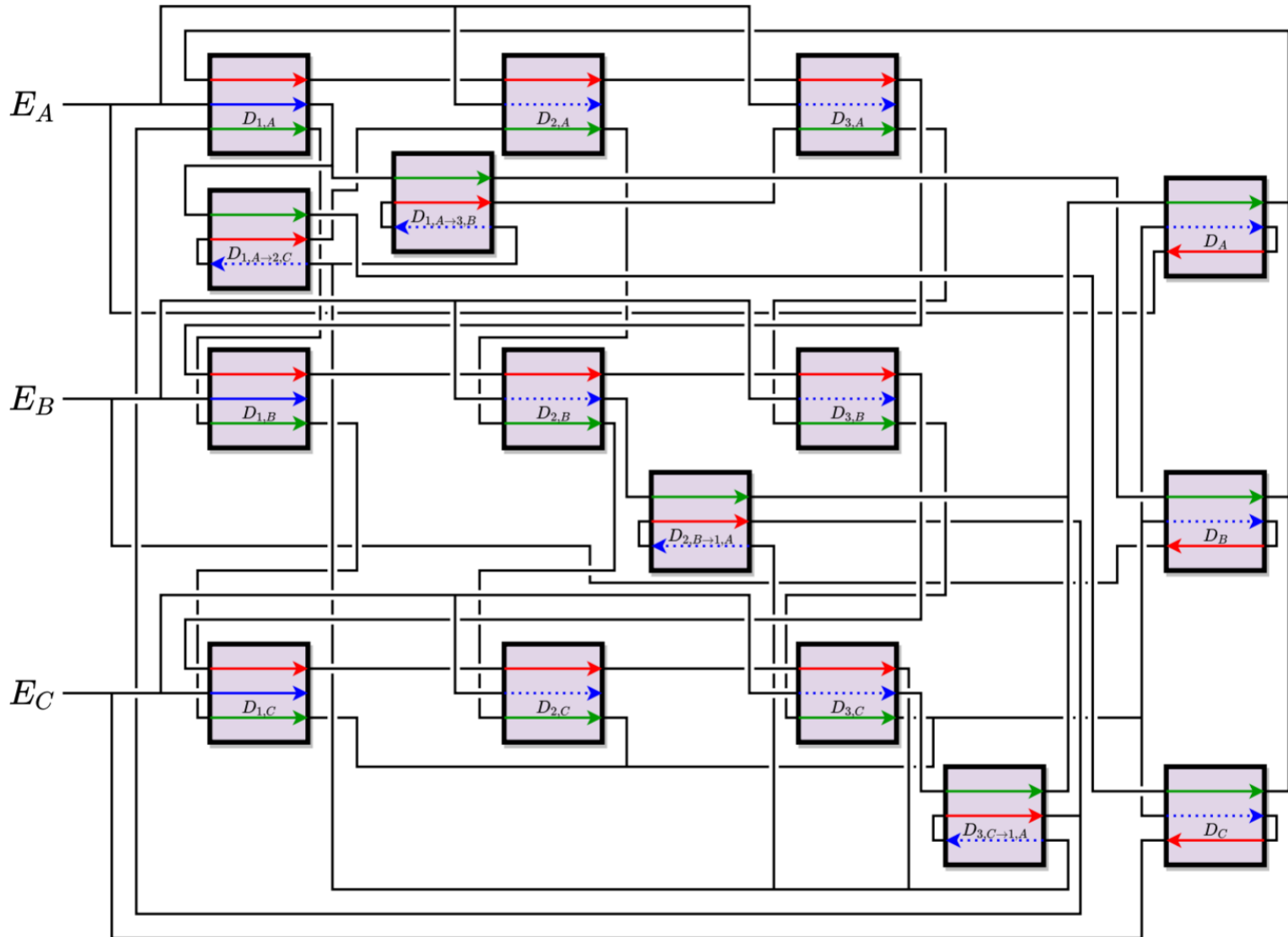
# Super Mario Odyssey is PSPACE-complete

[Ani, Bosboom, Demaine, Diomidov, Hendrickson, Lynch 2020]



# Any Door Gadget Simulates All Gadgets

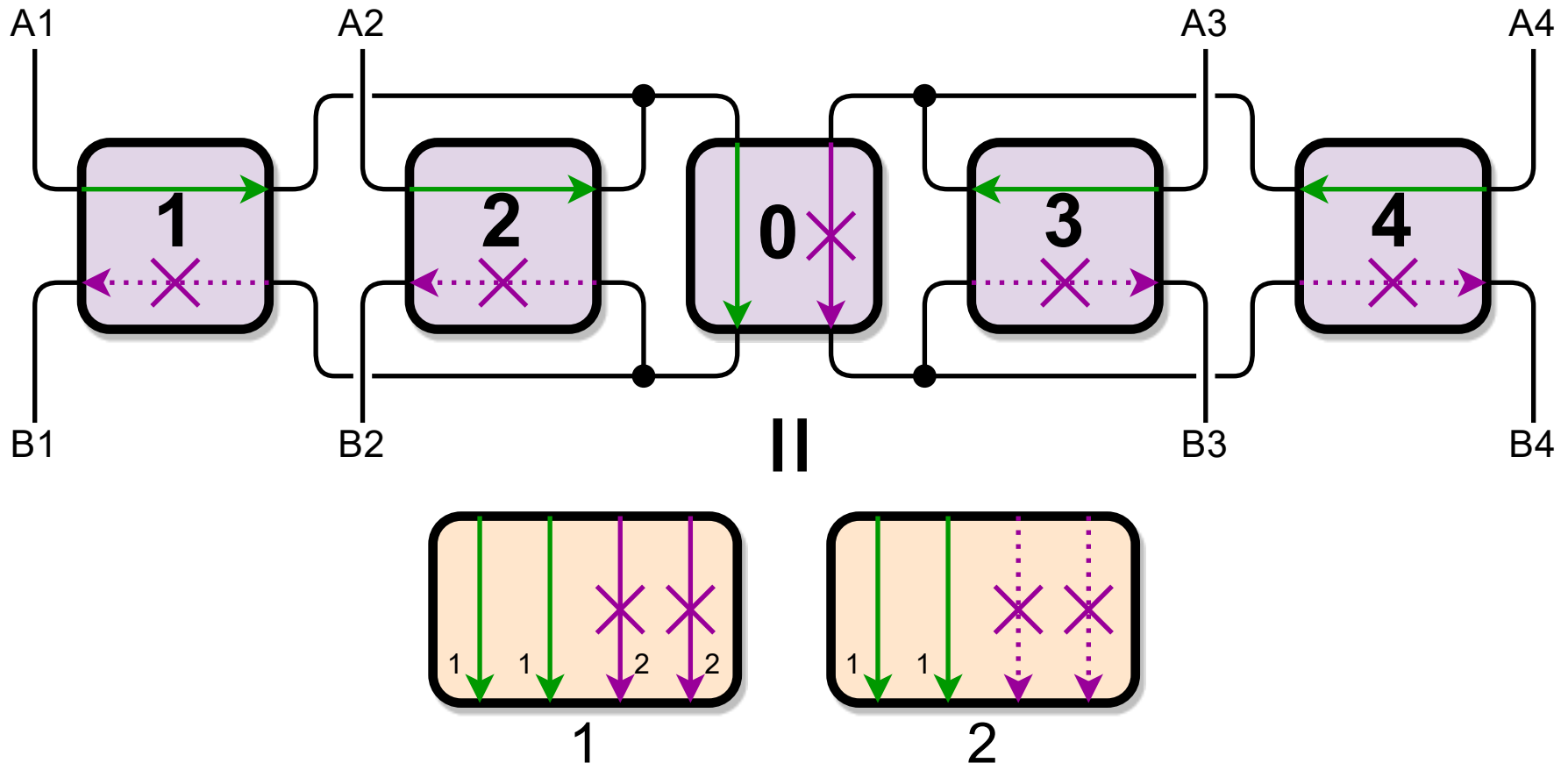
[Ani, Bosboom, Demaine,  
Diomidov, Hendrickson,  
Lynch 2020]



# Self-Closing Door Expansion

[Ani, Bosboom, Demaine, Diomidov, Hendrickson, Lynch 2020]

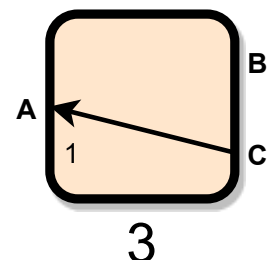
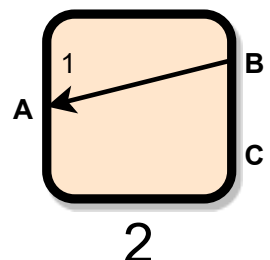
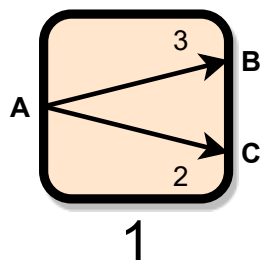
- Build self-closing door with several **open** and **self-close** tunnels





# Any Door Gadget Simulates All Gadgets

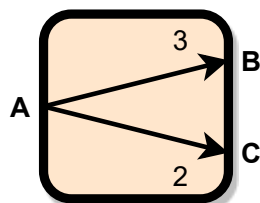
[Ani, Bosboom, Demaine,  
Diomidov, Hendrickson,  
Lynch 2020]



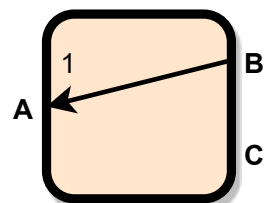
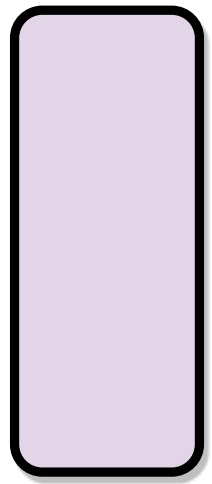


# Any Door Gadget Simulates All Gadgets

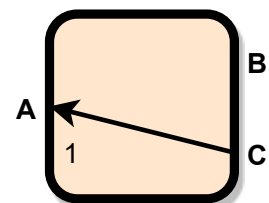
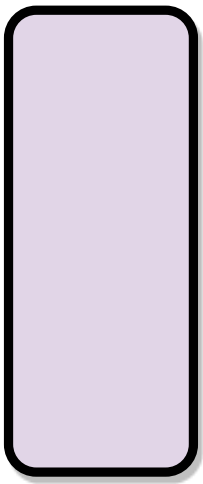
[Ani, Bosboom, Demaine,  
Diomidov, Hendrickson,  
Lynch 2020]



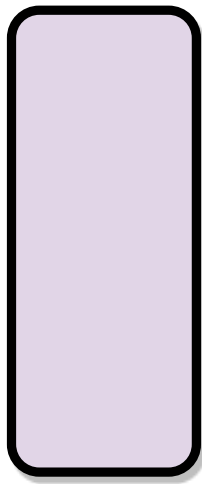
1



2



3



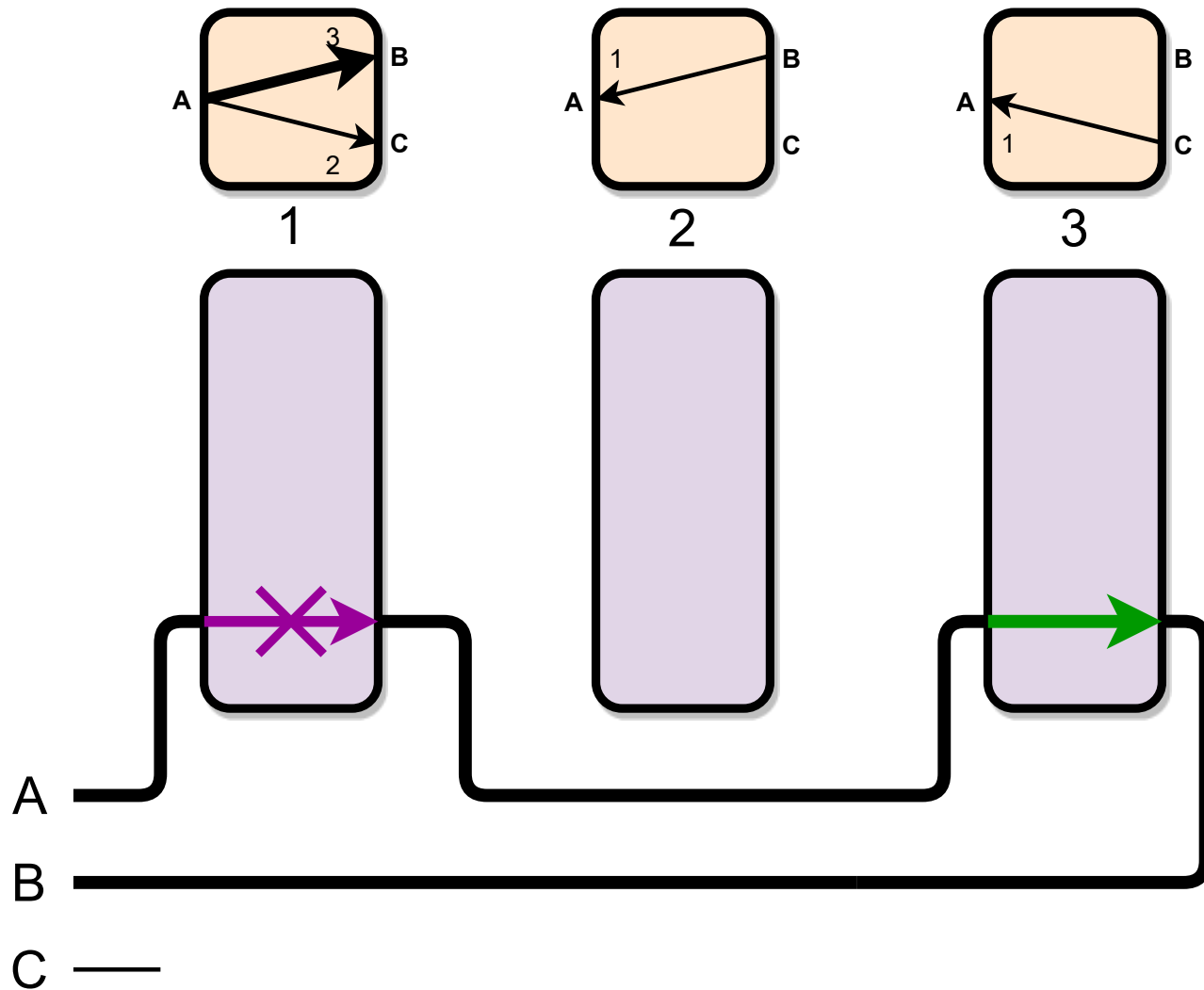
A —

B —

C —

# Any Door Gadget Simulates All Gadgets

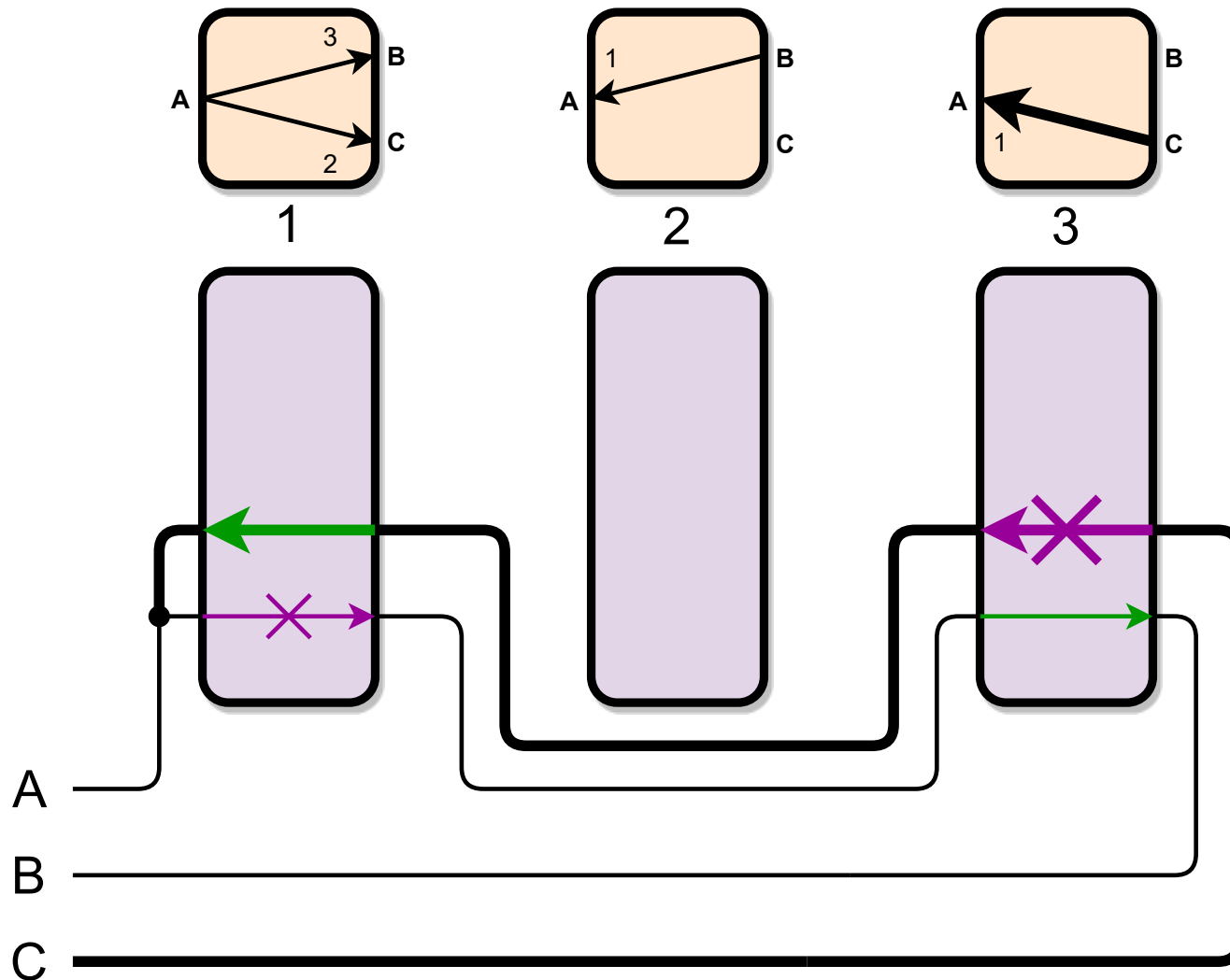
[Ani, Bosboom, Demaine,  
Diomidov, Hendrickson,  
Lynch 2020]





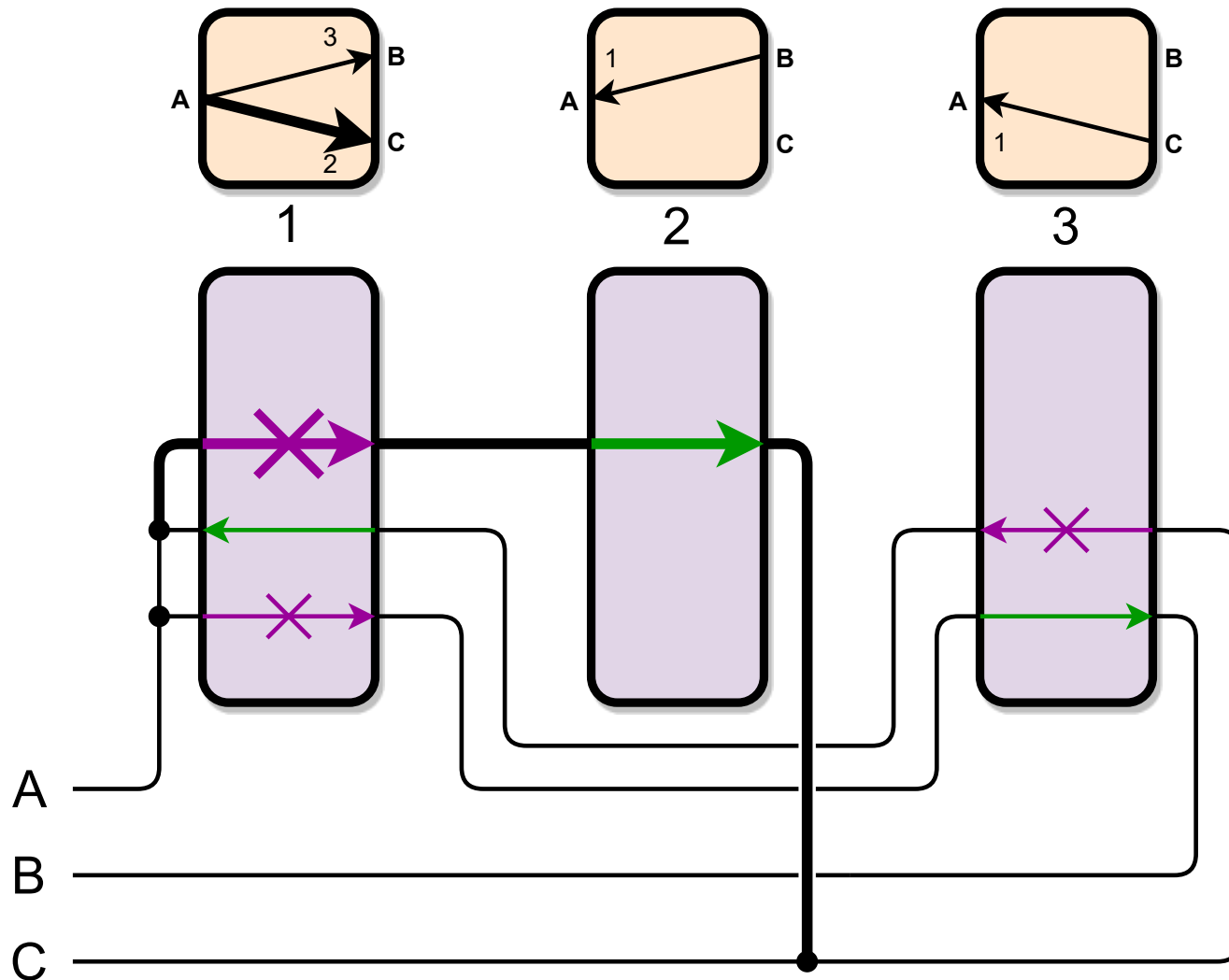
# Any Door Gadget Simulates All Gadgets

[Ani, Bosboom, Demaine,  
Diomidov, Hendrickson,  
Lynch 2020]



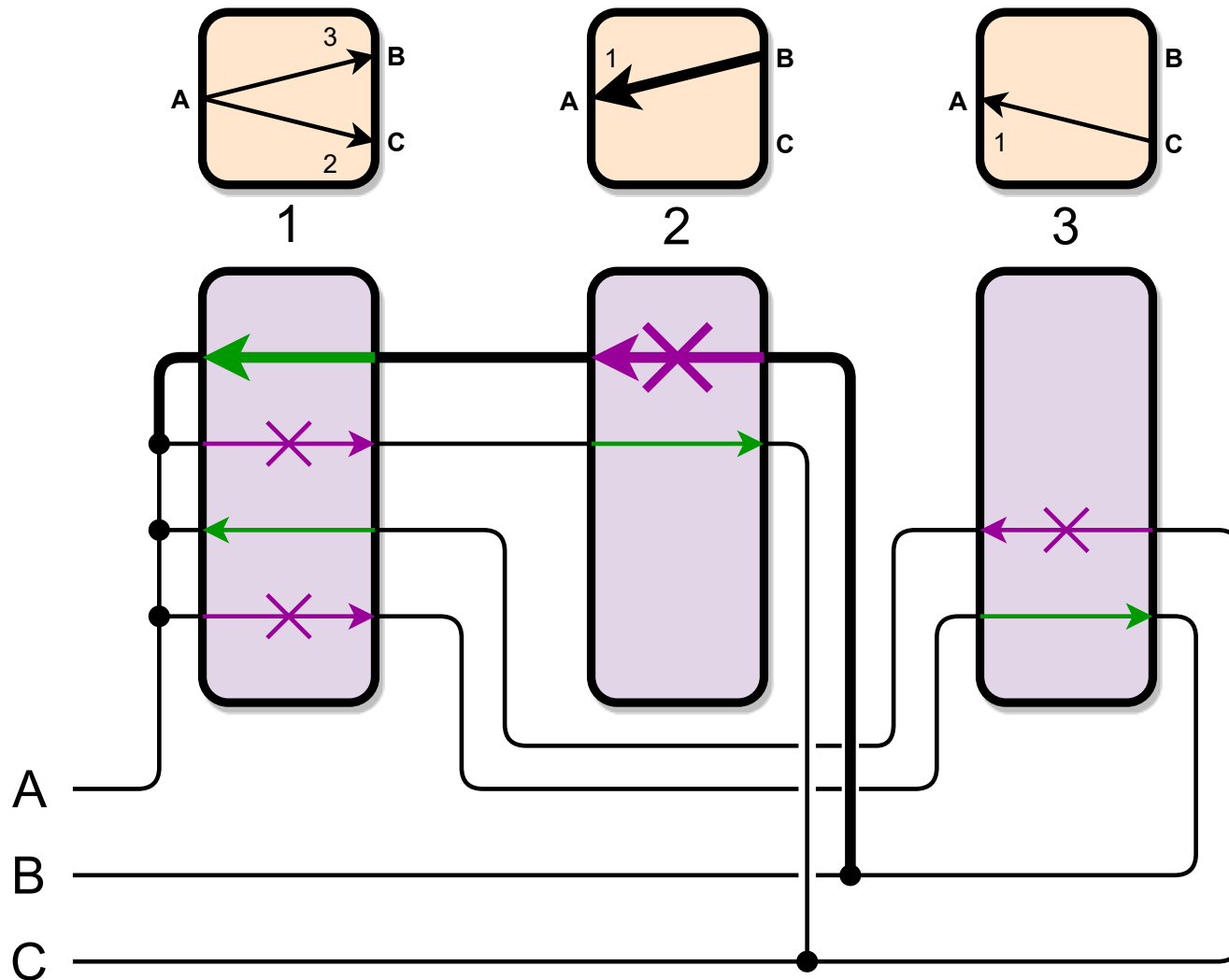
# Any Door Gadget Simulates All Gadgets

[Ani, Bosboom, Demaine,  
Diomidov, Hendrickson,  
Lynch 2020]



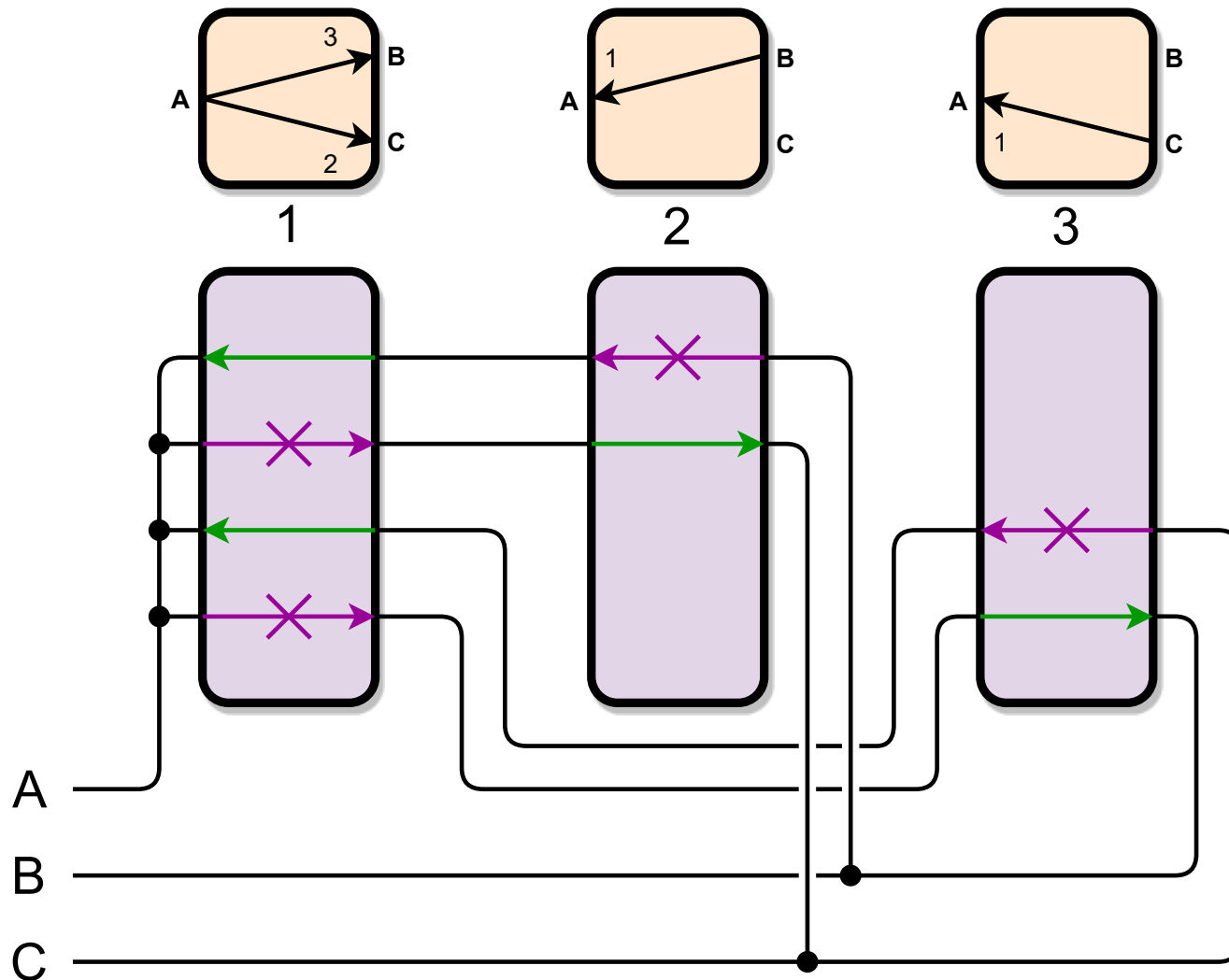
# Any Door Gadget Simulates All Gadgets

[Ani, Bosboom, Demaine,  
Diomidov, Hendrickson,  
Lynch 2020]



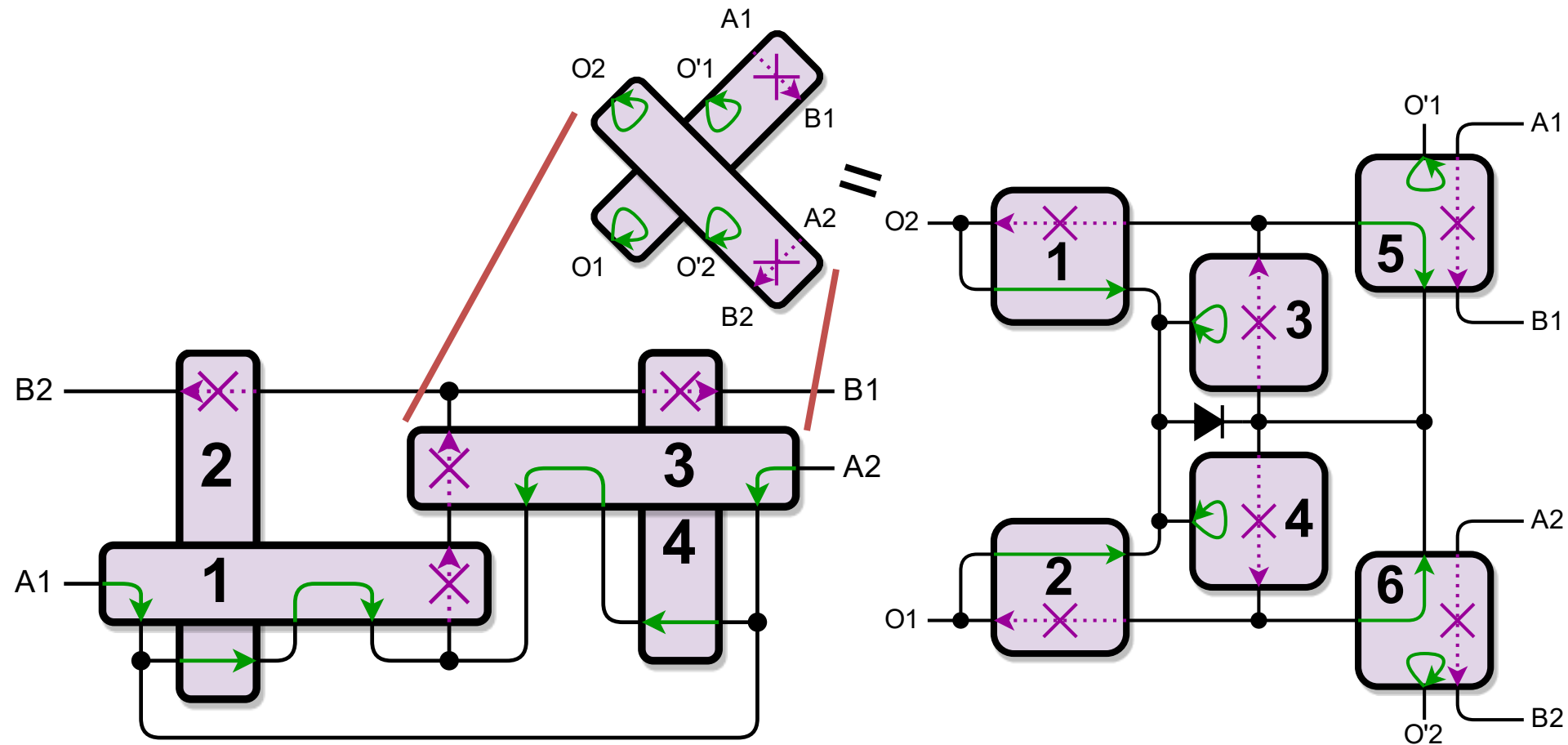
# Any Door Gadget Simulates All Gadgets

[Ani, Bosboom, Demaine,  
Diomidov, Hendrickson,  
Lynch 2020]

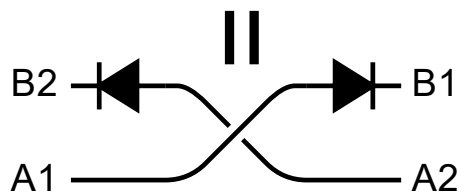


# Self-Closing Door Crossover

[Ani, Bosboom, Demaine, Diomidov, Hendrickson, Lynch 2020]



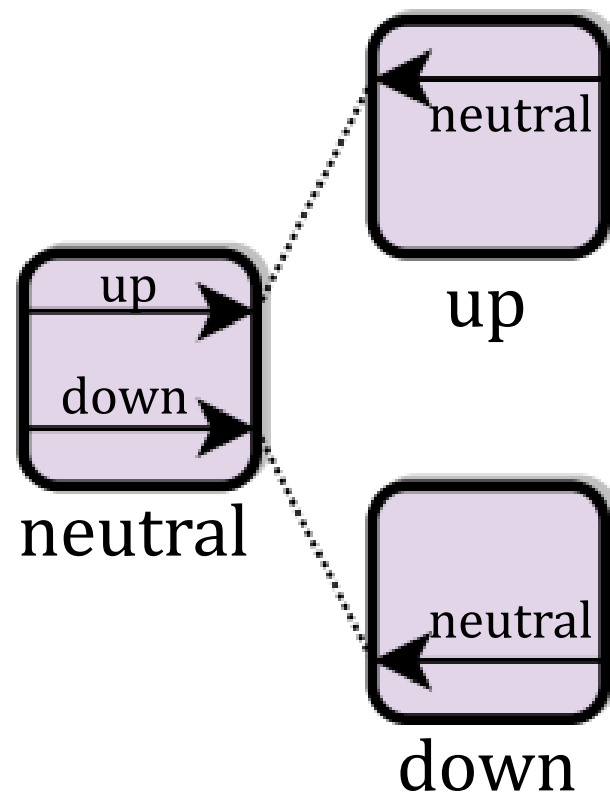
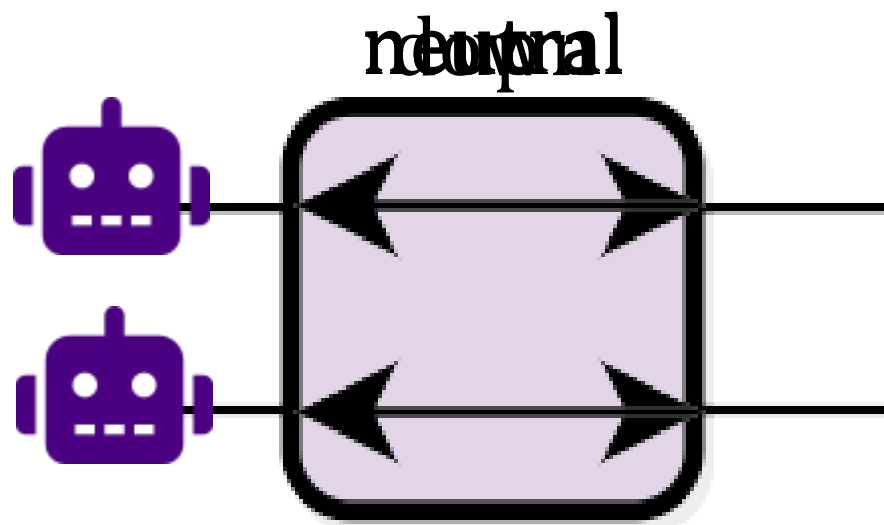
crossover





# Locking 2-Toggle

[Demaine,  
Hendrickson,  
Lynch 2020]

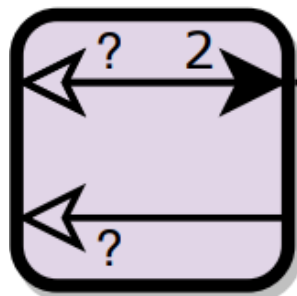




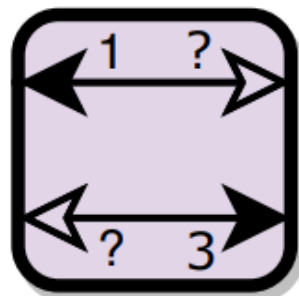
# Interacting-Tunnel Reversible Deterministic Tunnel Gadget Simulates Locking 2-Toggle

[Demaine,  
Hendrickson,  
Lynch 2020]

1



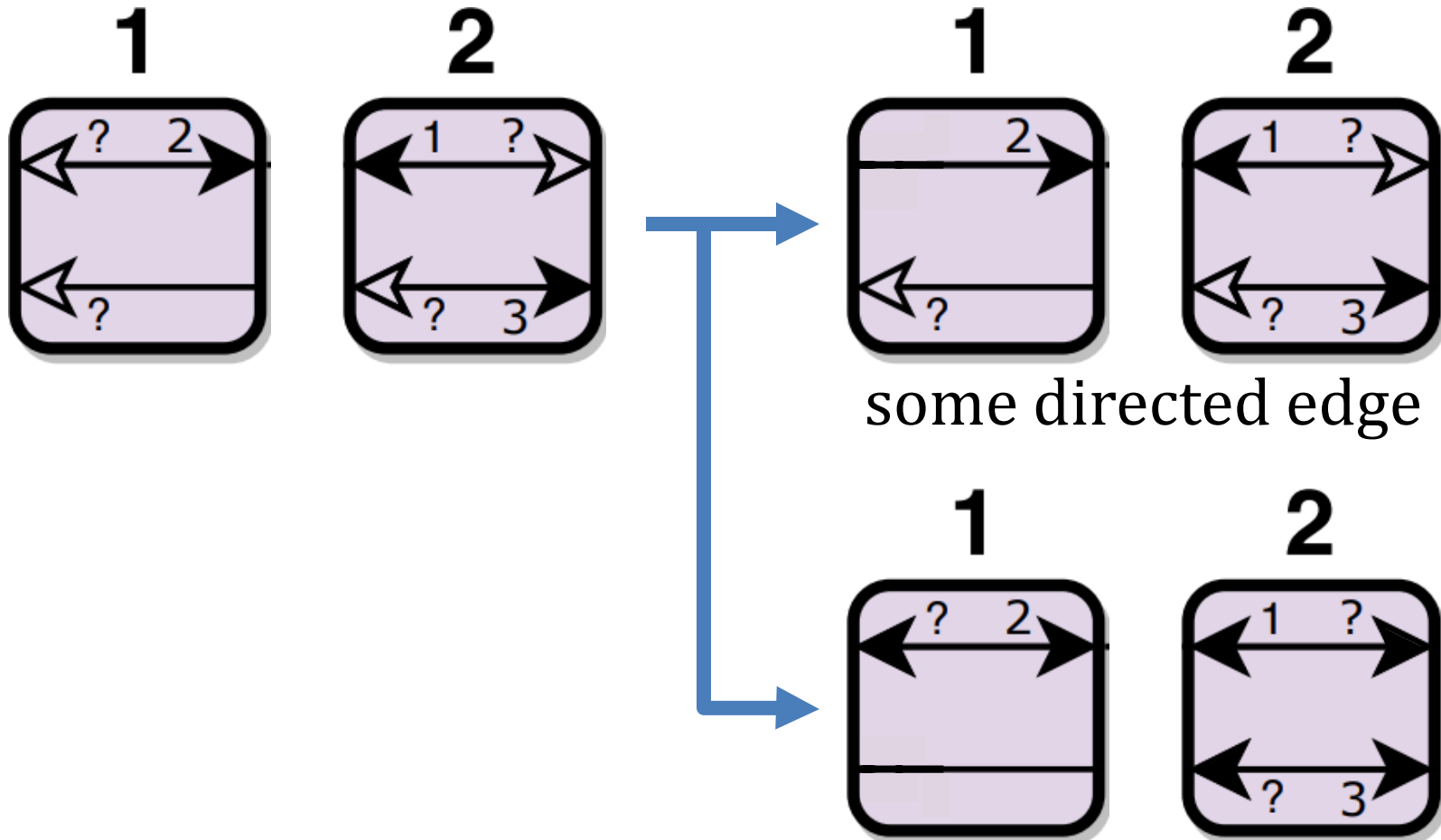
2





# Interacting-Tunnel Reversible Deterministic Tunnel Gadget Simulates Locking 2-Toggle

[Demaine,  
Hendrickson,  
Lynch 2020]

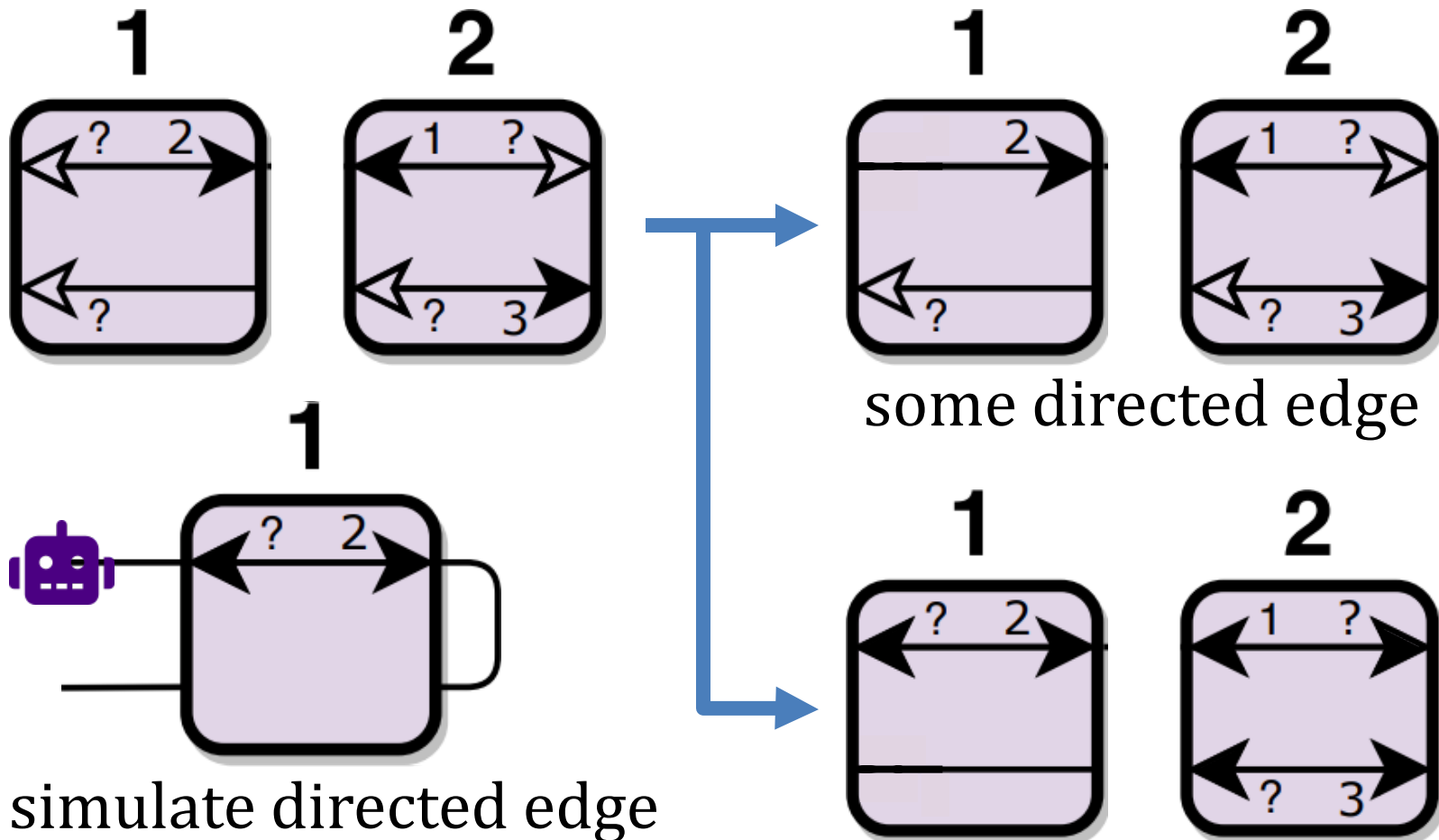






# Interacting-Tunnel Reversible Deterministic Tunnel Gadget Simulates Locking 2-Toggle

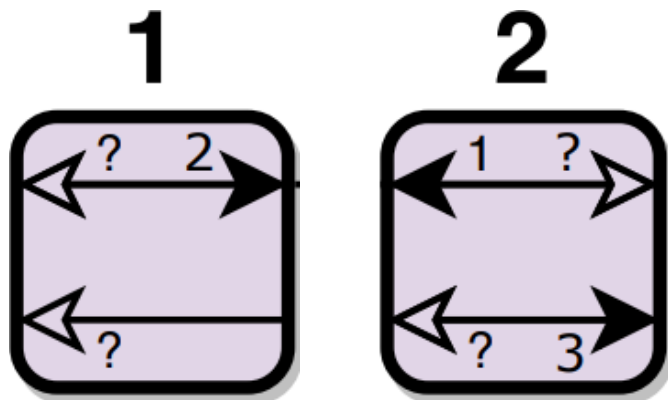
[Demaine,  
Hendrickson,  
Lynch 2020]



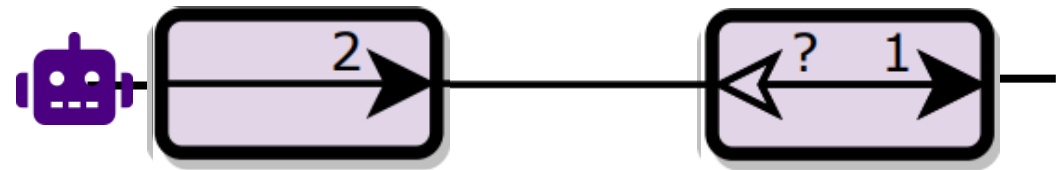


# Interacting-Tunnel Reversible Deterministic Tunnel Gadget Simulates Locking 2-Toggle

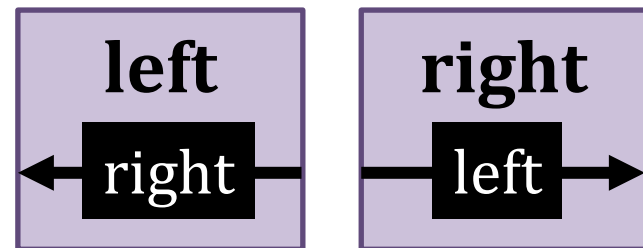
[Demaine,  
Hendrickson,  
Lynch 2020]



directed edge



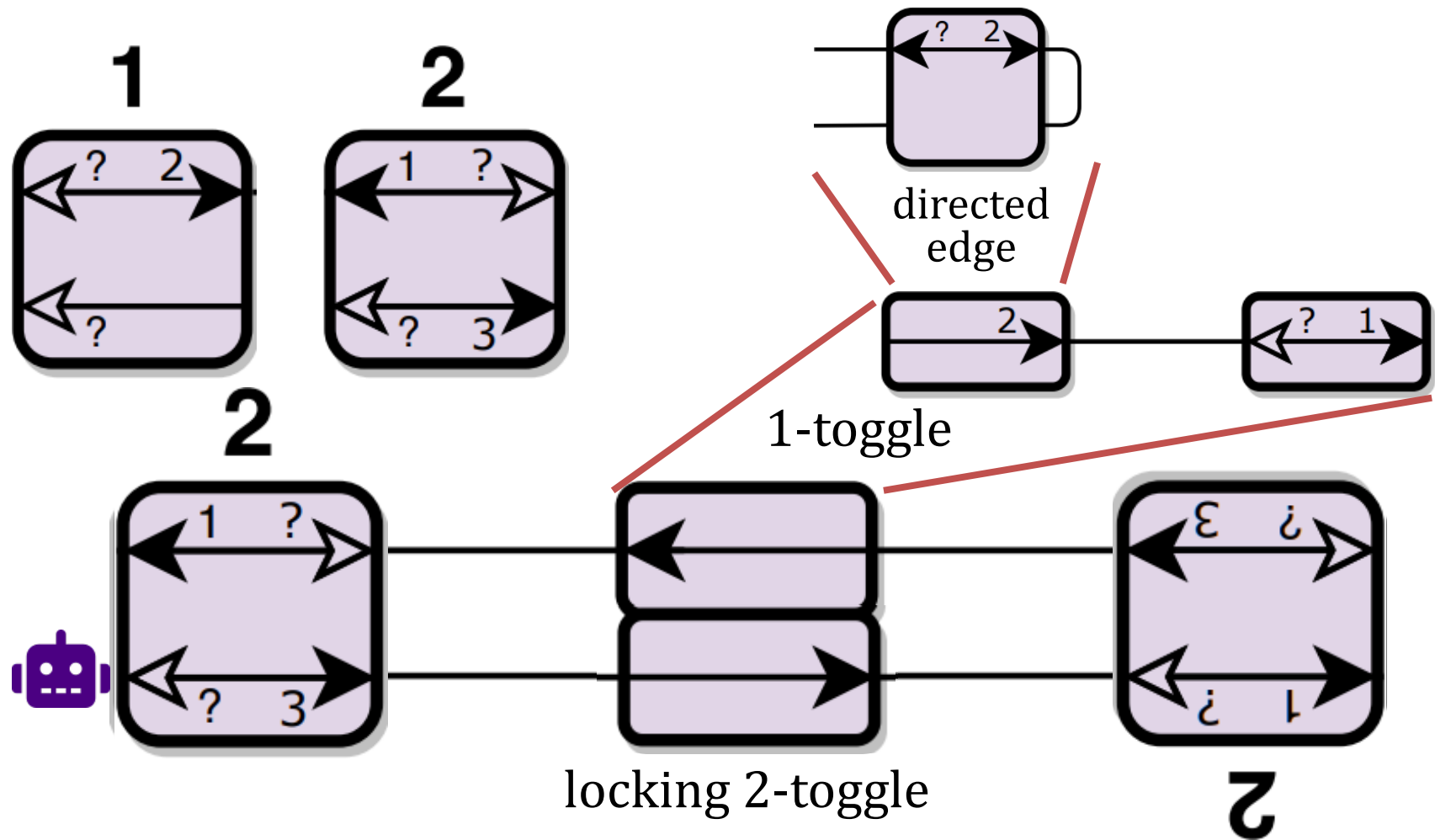
1-Toggle





# Interacting-Tunnel Reversible Deterministic Tunnel Gadget Simulates Locking 2-Toggle

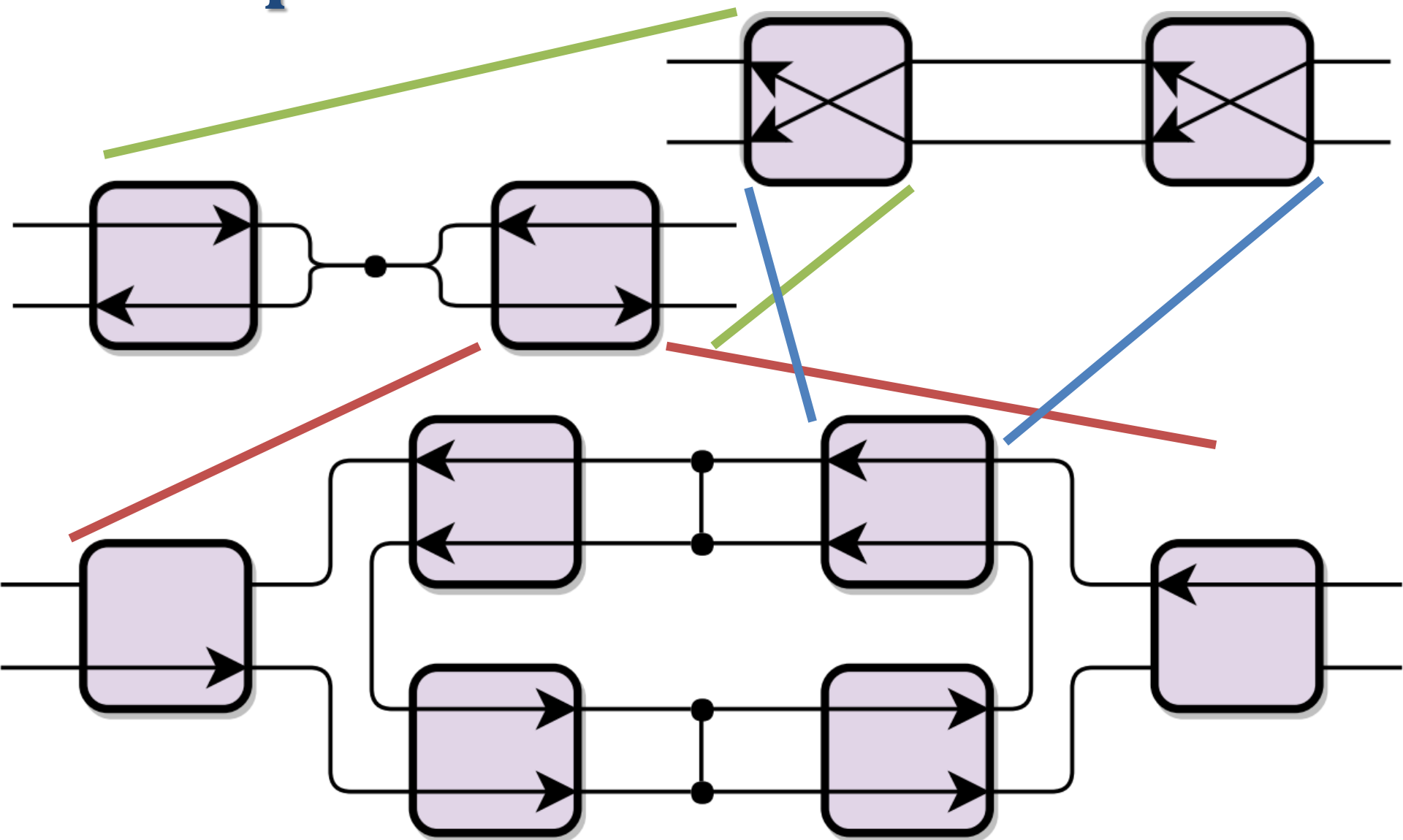
[Demaine,  
Hendrickson,  
Lynch 2020]





# Planar Locking 2-Toggles are Equivalent

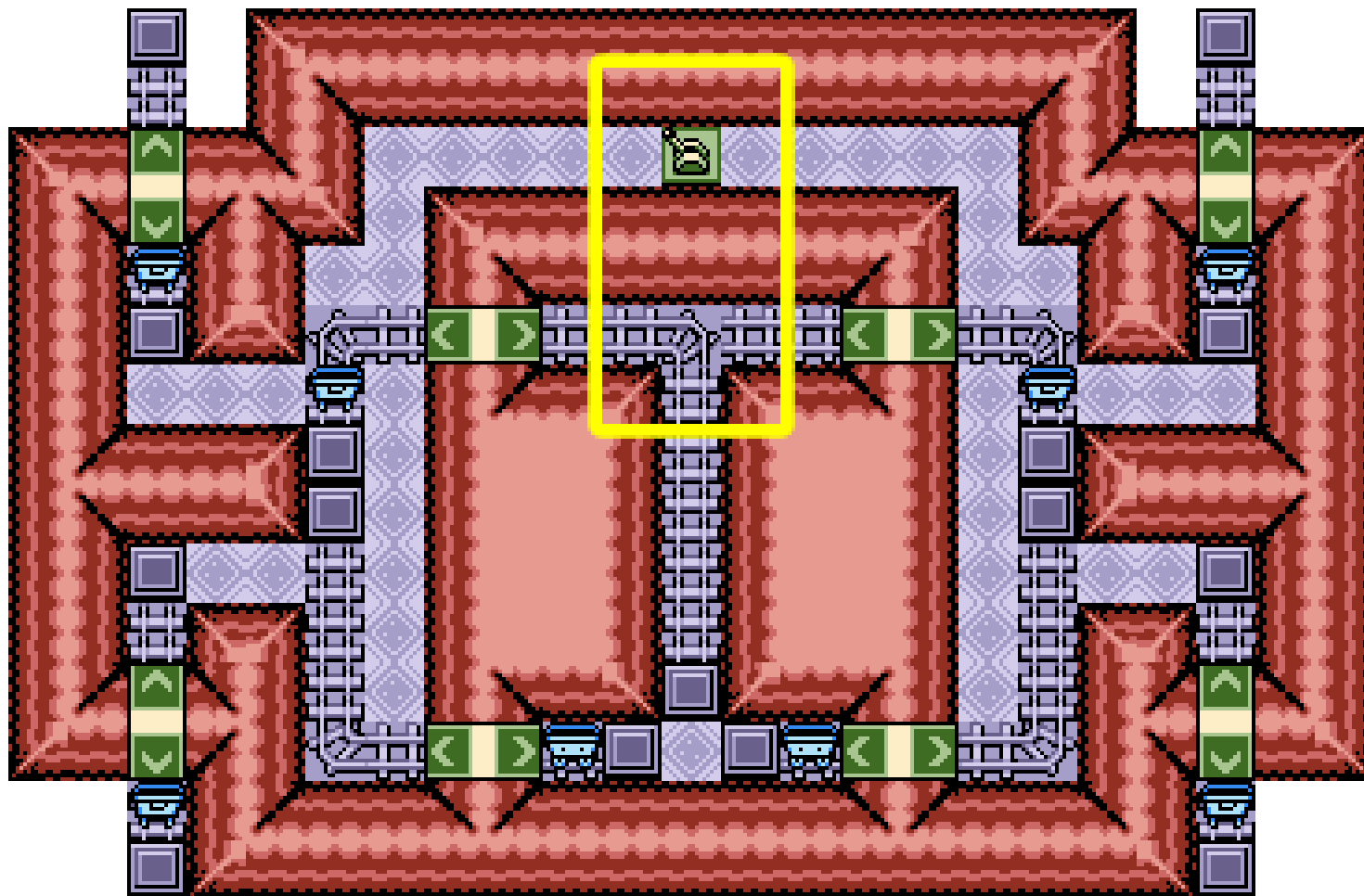
[Demaine, Hendrickson, Lynch 2020]





# Legend of Zelda: Oracle of Ages/Seasons Minecarts are PSPACE-complete

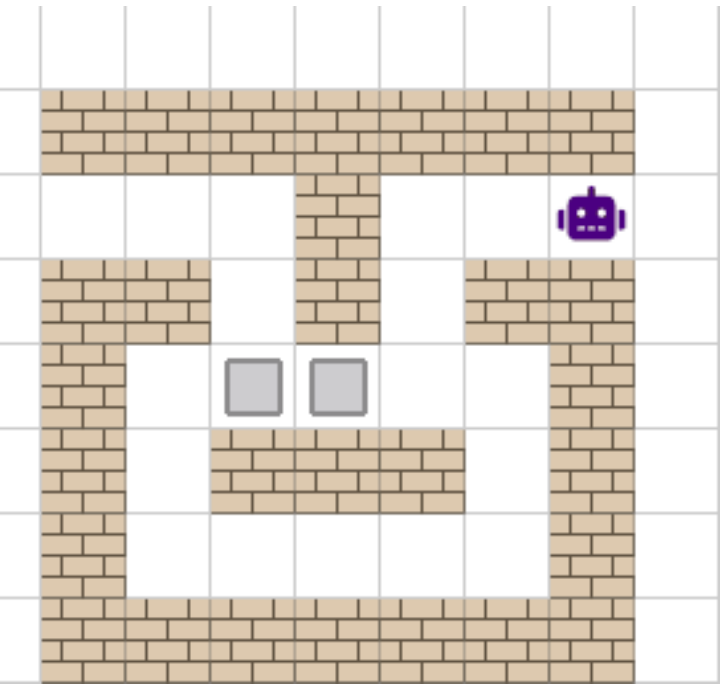
[Bosboom, Brunner,  
Coulombe, Demaine,  
Hendrickson, Lynch,  
Najt 2022]



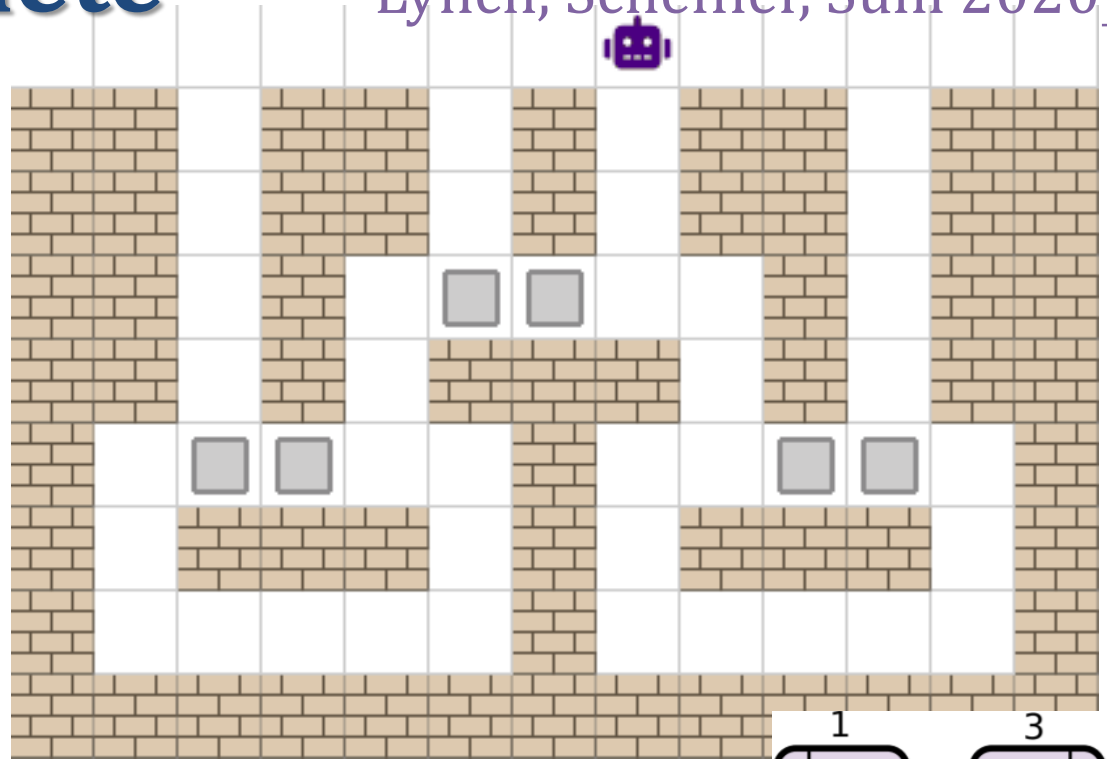
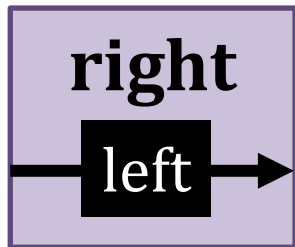
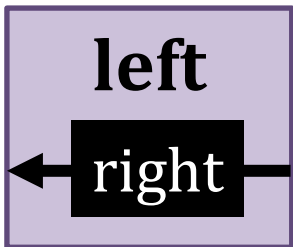
locking  
2-toggle

# Pull?-2F(G) is PSPACE-complete

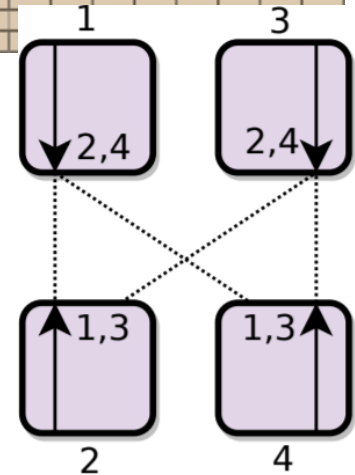
[Ani, Asif, Demaine, Diomidov, Hendrickson, Lynch, Scheffler, Suhl 2020]



**1-Toggle**



**Nondeterministic Locking 2-Toggle**



# Pulling $1 \times 1$ Blocks Complexity

Name	Pull	Force	Fixed	Gravity	Complexity
PULL?- $k$ F	$k \geq 1$	no	yes	no	PSPACE-complete
PULL?-*F	$\infty$	no	yes	no	PSPACE-complete
PULL!- $k$ F	$k \geq 1$	yes	yes	no	PSPACE-complete
PULL!-*F	$\infty$	yes	yes	no	PSPACE-complete
PULL?-1FG	1	no	yes	yes	NP-hard
PULL?-1WG	1	no	yes, thin	yes	PSPACE-complete
PULL?- $k$ FG	$k \geq 2$	no	yes	yes	PSPACE-complete
PULL?-*FG	$\infty$	no	yes	yes	PSPACE-complete
PULL!- $k$ FG	$k \geq 1$	yes	yes	yes	PSPACE-complete
PULL!-*FG	$\infty$	yes	yes	yes	PSPACE-complete

[Ani, Asif, Demaine, Diomidov,  
Hendrickson, Lynch, Scheffler, Suhl 2020]



# Pivoting Modular Robots:

**M-Blocks** [Sung, Bern, Romanishin, Rus 2015]

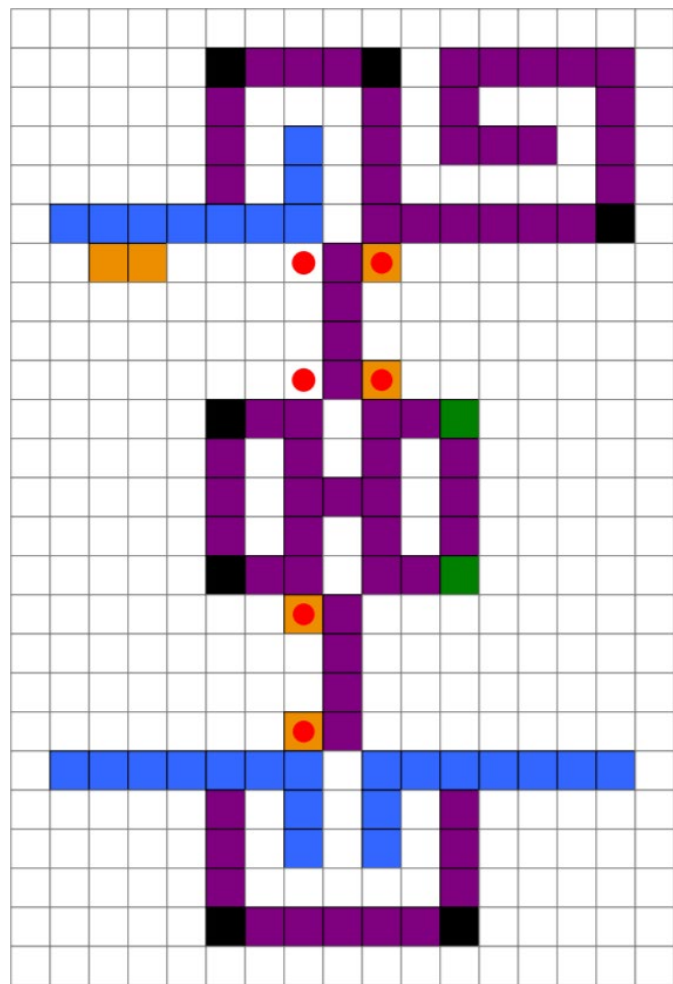
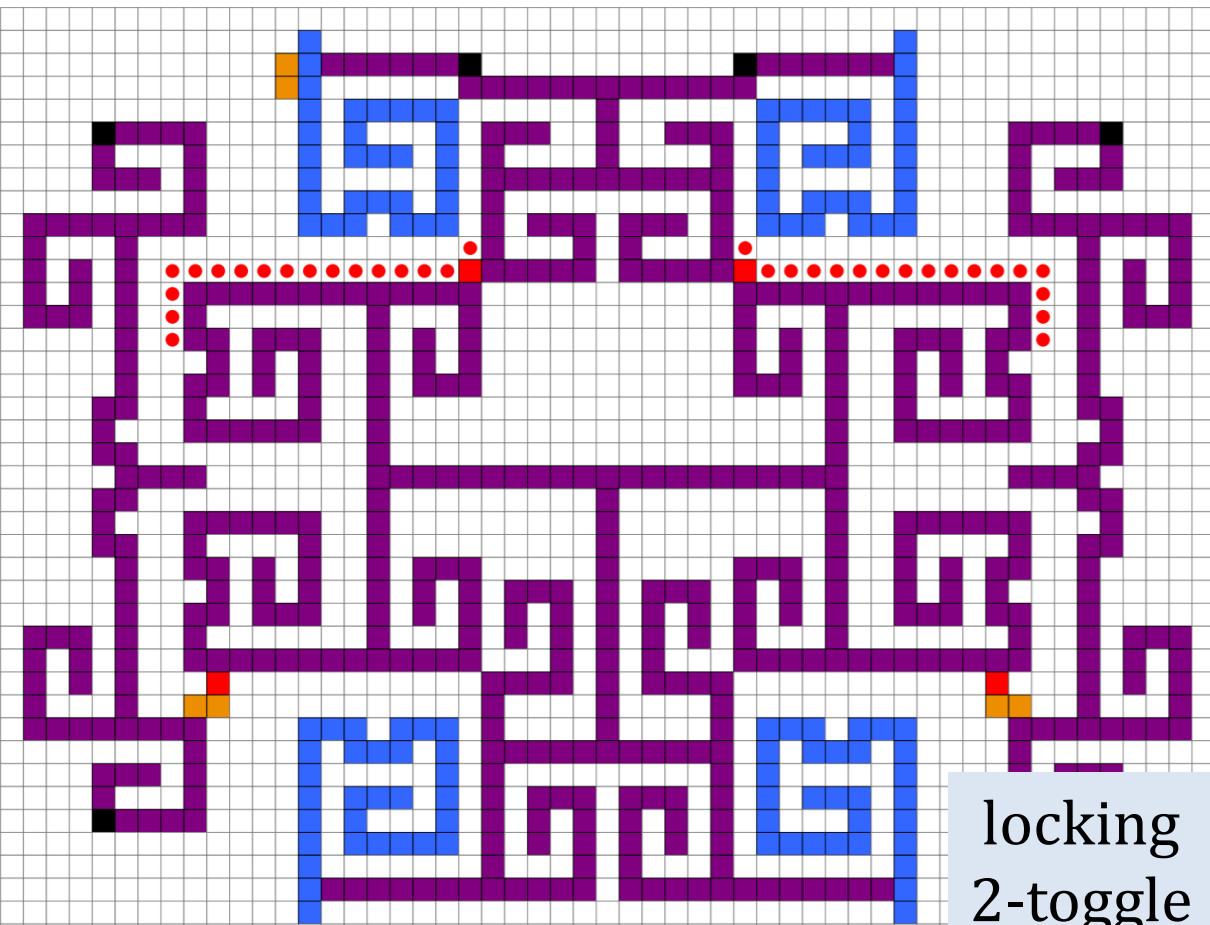
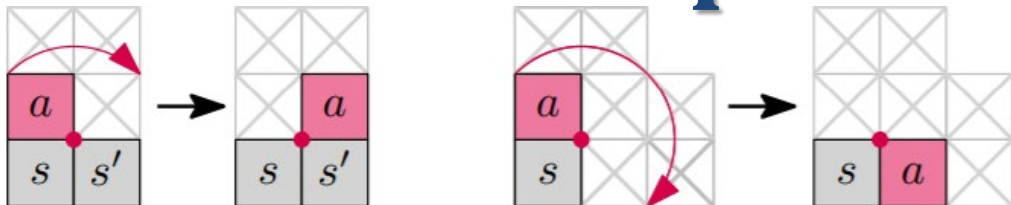






# Pivoting Squares is PSPACE-complete

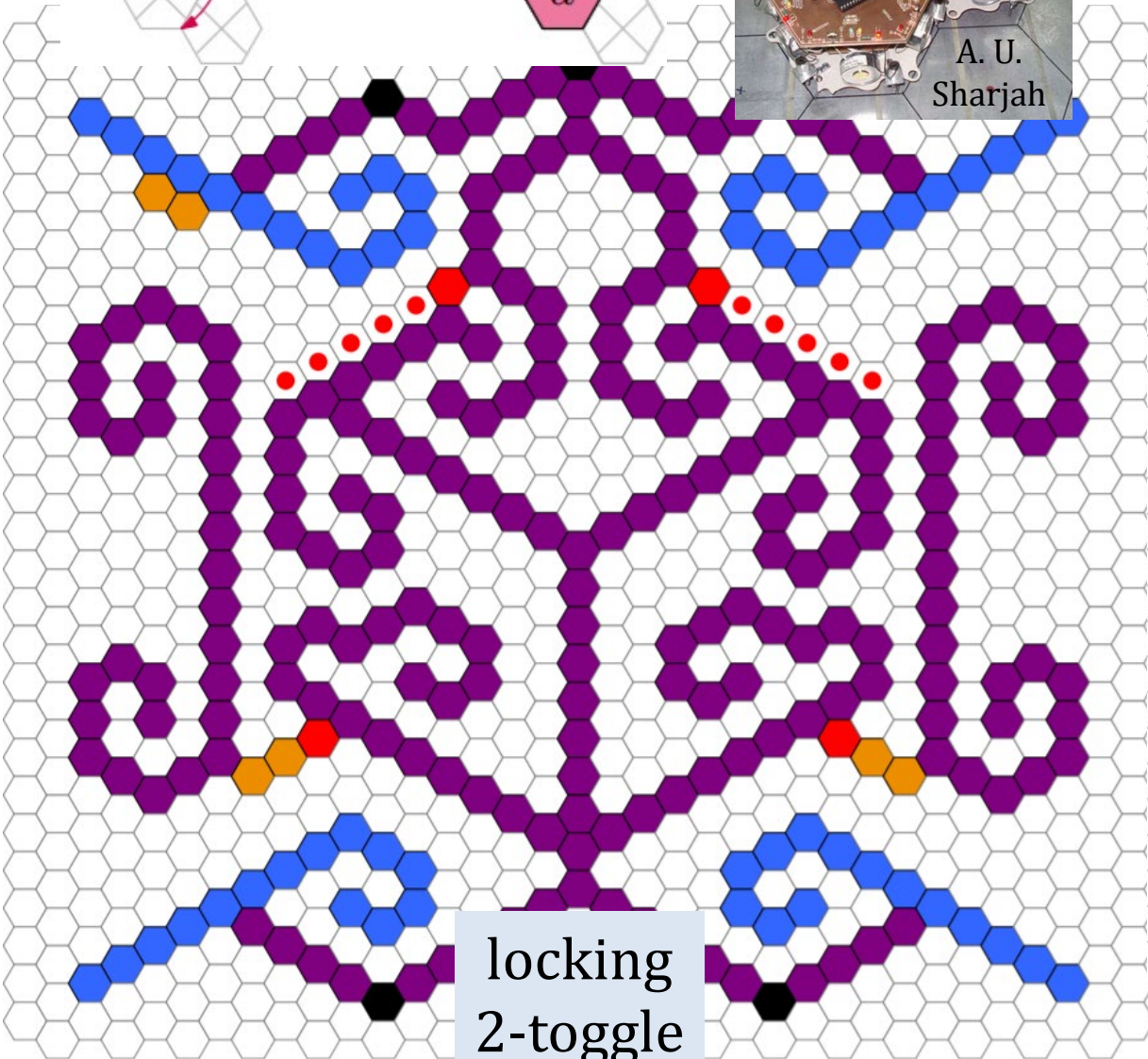
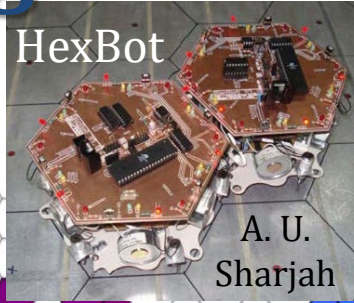
[Akitaya, Demaine, Gonczi,  
Hendrickson, Hesterberg,  
Korman, Korten, Lynch,  
Parada, Sacristan 2021]



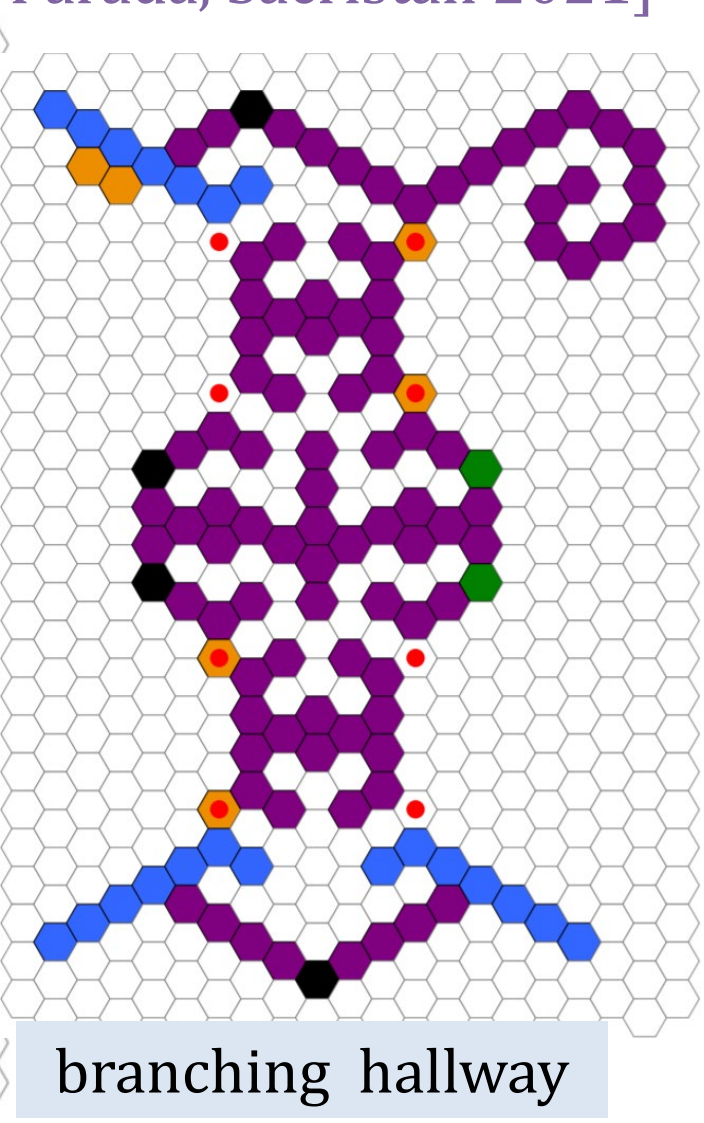
branching hallway

# Pivoting Hexagons

[Akitaya, Demaine, Gonczi, Hendrickson, Hesterberg, Korman, Korten, Lynch, Parada, Sacristan 2021]



locking  
2-toggle



branching hallway

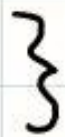
# 2-State 2-Tunnel Reversible Deterministic Gadgets

[Demaine, Grosf, Lynch, Rudoy 2018]

Tunnel types:



tripwire: always bitraversable & traversal flips state



} lock: bitraversable in one state not in other



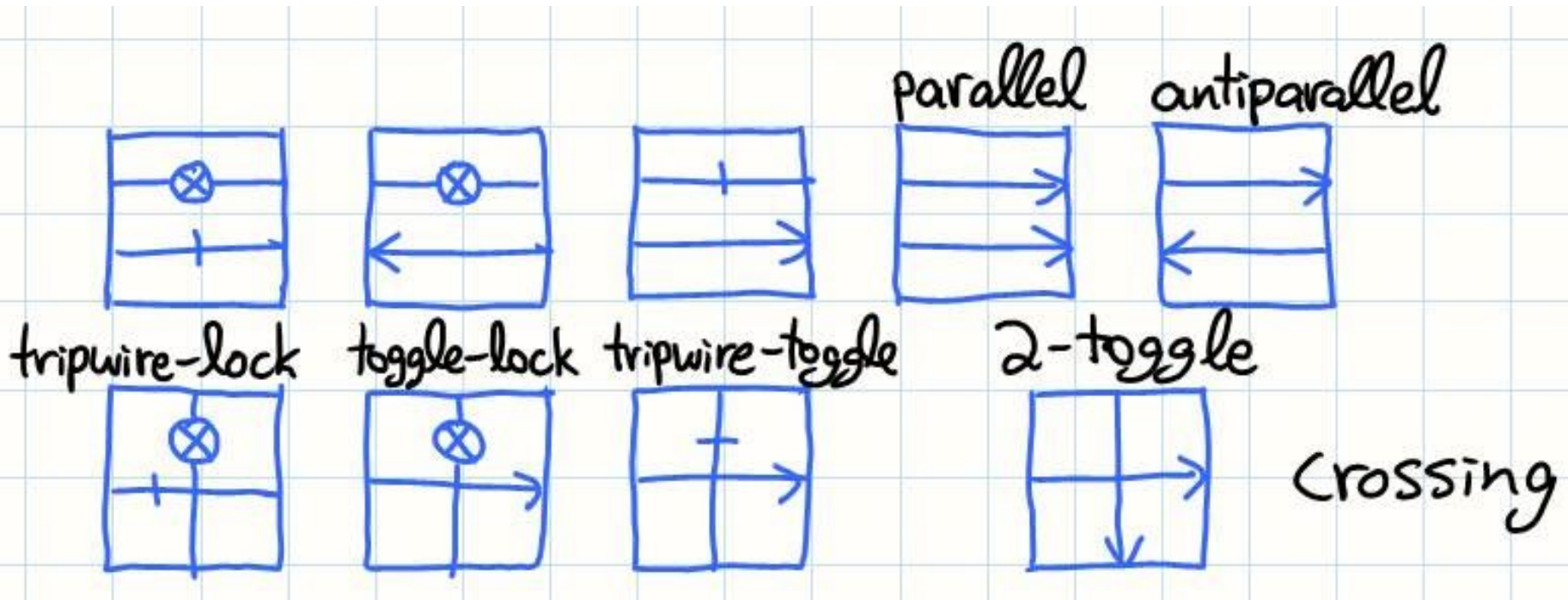
toggle: traversal in one direction & toggles state + direction



} trivial: always/never bitraversable

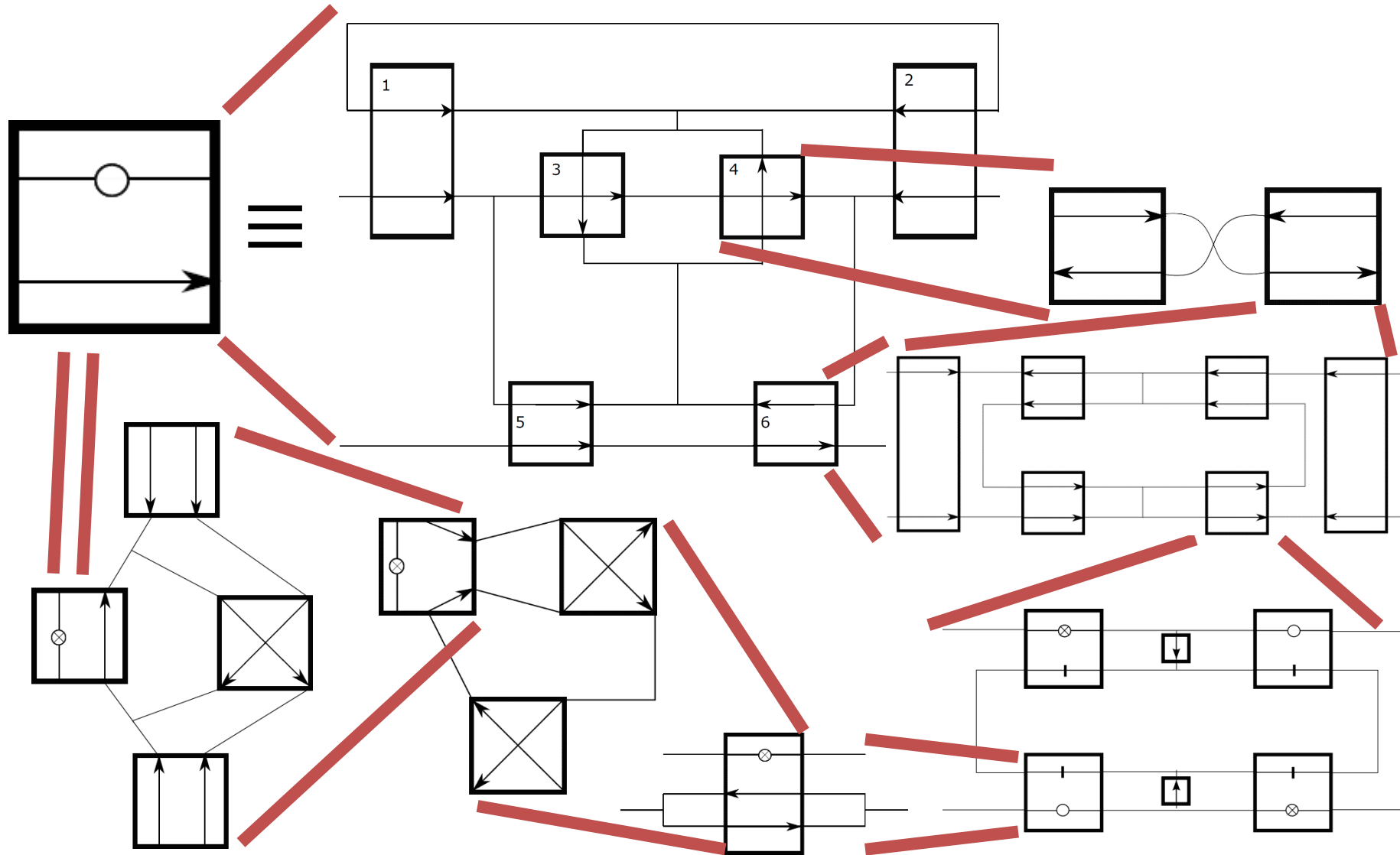
# 2-State 2-Tunnel Reversible Deterministic Gadgets

[Demaine, Grosof, Lynch, Rudoy 2018]



# Everything Simulates Everything

[Demaine, Grosz, Lynch, Rudoy 2018]

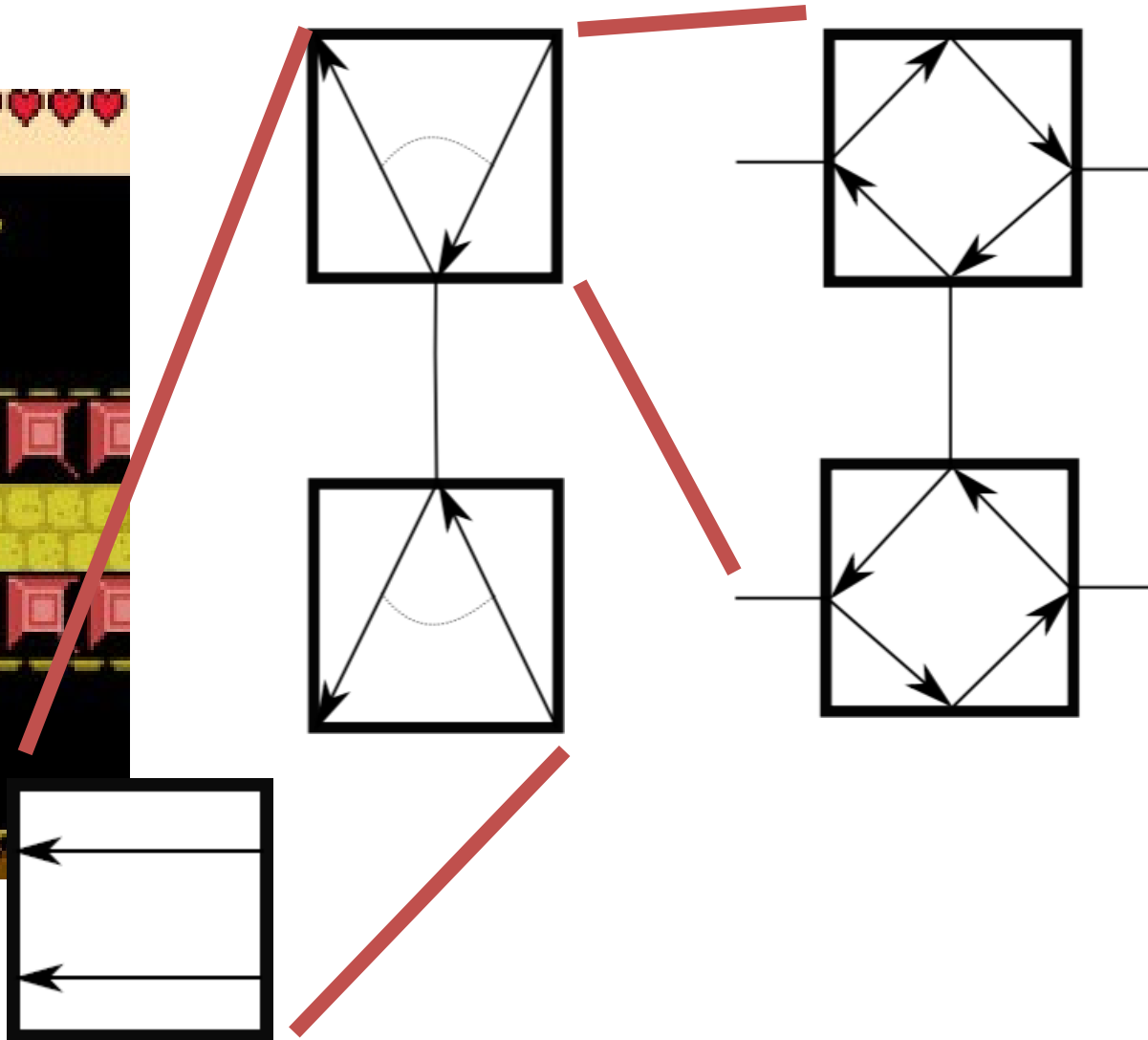


# 4-Spinners are PSPACE-hard

[Demaine, Grosof, Lynch, Rudoy 2018]

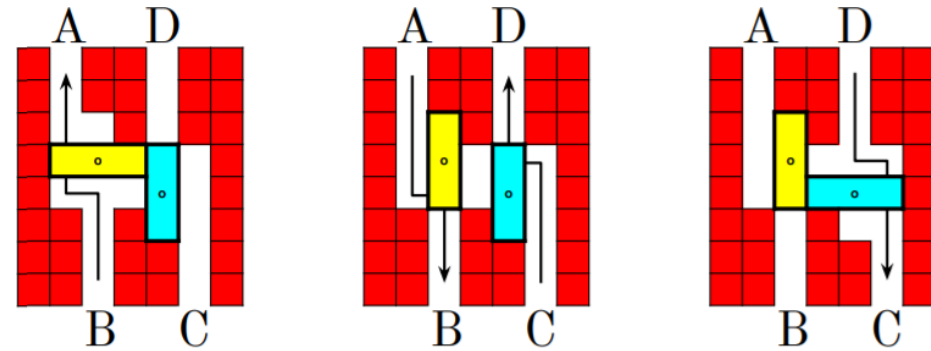
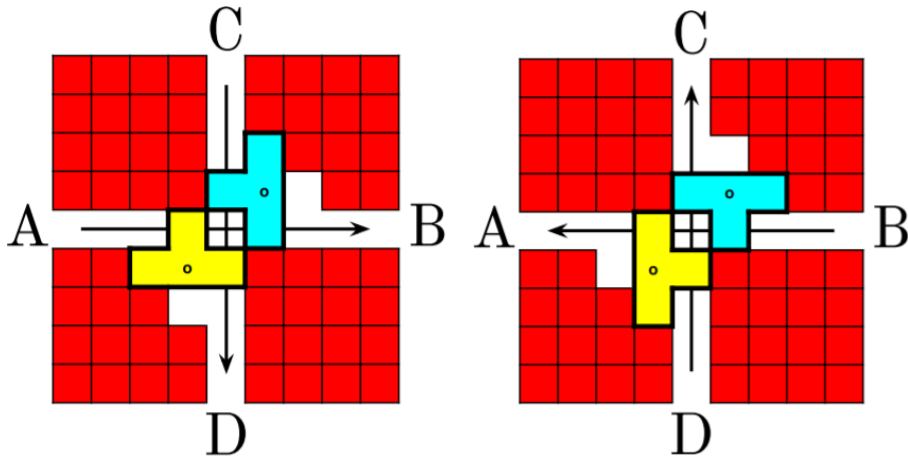


The Legend of Zelda:  
Oracle of Seasons

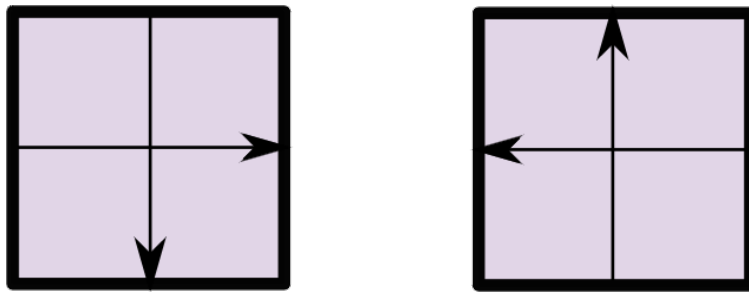


# Turnstiles are PSPACE-complete

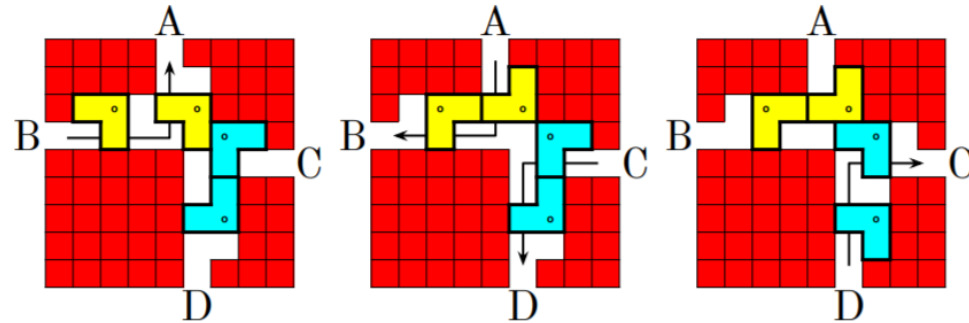
[Greenblatt, Hernandez, Hearn, Hou, Ito, Kang, Williams, Winslow 2021]



Antiparallel Locking 2-Toggle



Crossing 2-Toggle



Parallel Locking 2-Toggle

Applications: Kwirk, Pokémon Ruby, Super Mario Odyssey

# Pushing $1 \times 1$ Blocks Complexity

Name	Push	Fixed	Slide	Goal	Complexity	Reference
PUSH- $k$	$k \geq 1$	no	min	path	NP-hard	D, D, O'Rourke 2000
PUSH-*	$\infty$	no	min	path	NP-hard	Hoffmann 2000
PUSHPUSH- $k$	$k \geq 1$	no	max	path	PSPACE-complete	D, Hoffmann, Holzer 2004
PUSHPUSH-*	$\infty$	no	max	path	NP-hard	Hoffmann 2000
PUSH-1F	1	yes	min	path	NP-hard	DDO 2000
SOKOBAN	1	yes	min	storage	PSPACE-complete	Culberson 1998
PUSH- $k$ F	$k \geq 2$	yes	min	path	PSPACE-complete	D, Hearn, Hoffmann 2002
PUSH- $*$ F	$\infty$	yes	min	path	PSPACE-complete	Bremner, O'Rourke, Shermer 1994
PUSH- $k$ X	$k \geq 1$	no	min	simple path	NP-complete	D, Hoffmann 2001
PUSH- $*$ X	$\infty$	no	min	simple path	NP-complete	Hoffmann 2000

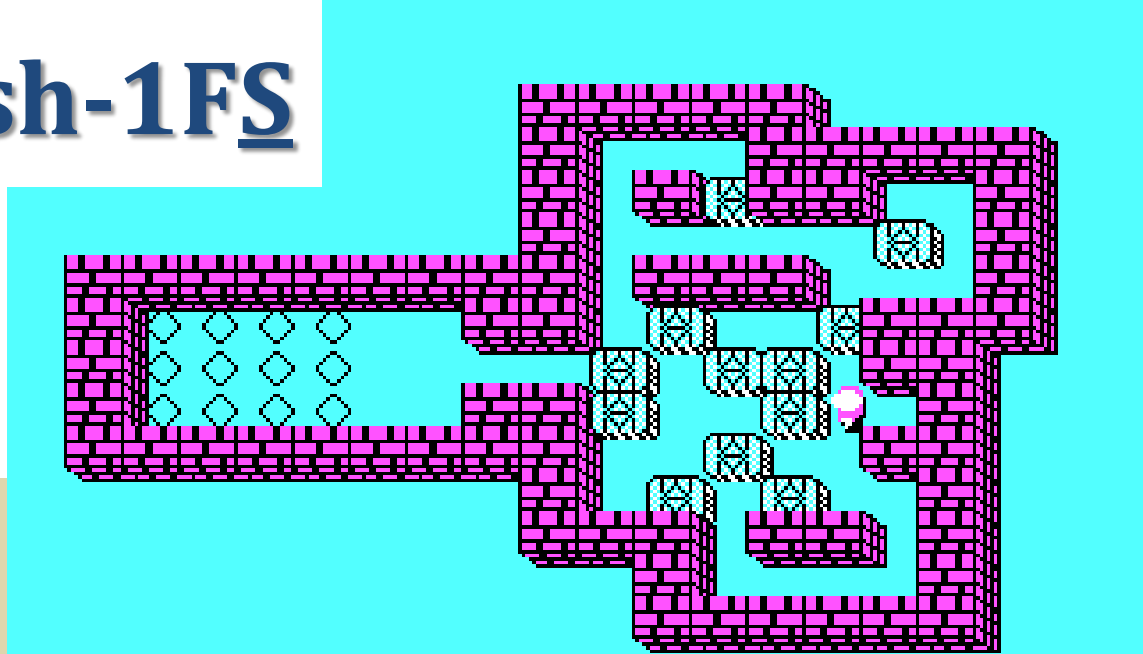
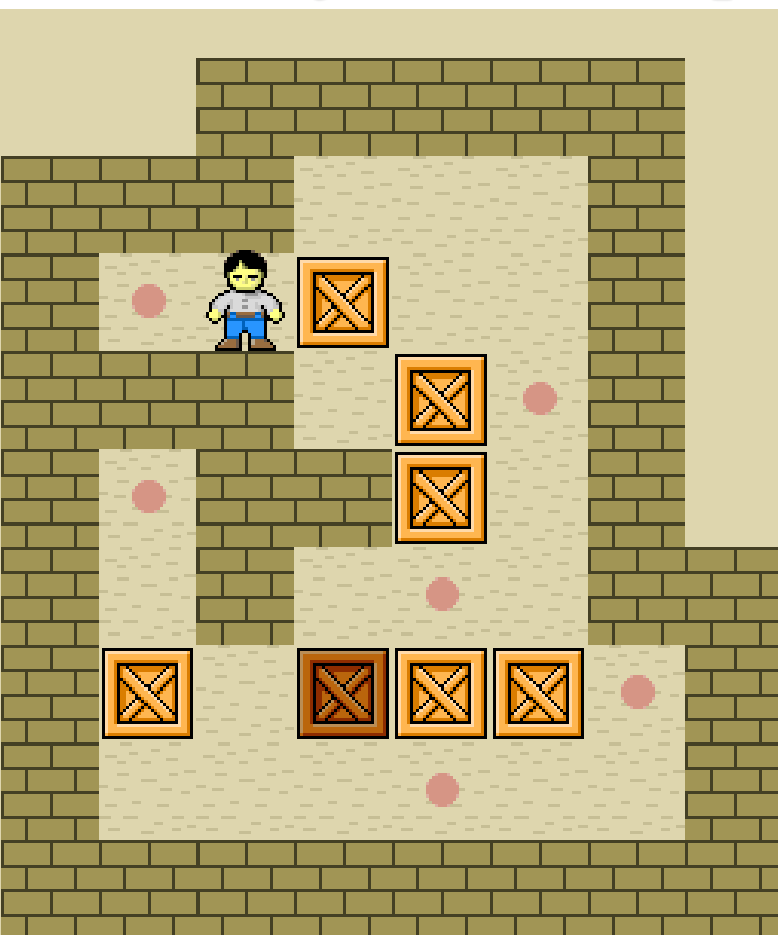


# Pushing $1 \times 1$ Blocks Complexity

Name	Push	Fixed	Slide	Goal	Complexity	Reference
PUSH- $k$	$k \geq 1$	no	min	path	NP-hard	D, D, O'Rourke 2000
PUSH-*	$\infty$	no	min	path	NP-hard	Hoffmann 2000
PUSHPUSH- $k$	$k \geq 1$	no	max	path	PSPACE-complete	D, Hoffmann, Holzer 2004
PUSHPUSH-*	$\infty$	no	max	path	NP-hard	Hoffmann 2000
PUSH-1F	1	yes	min	path	PSPACE-complete	Ani, Chung, D, Diomidov, Hendrickson, Lynch 2022
SOKOBAN	1	yes	min	storage	PSPACE-complete	Culberson 1998
PUSH- $k$ F	$k \geq 2$	yes	min	path	PSPACE-complete	D, Hearn, Hoffmann 2002
PUSH- $*$ F	$\infty$	yes	min	path	PSPACE-complete	Bremner, O'Rourke, Shermer 1994
PUSH- $k$ X	$k \geq 1$	no	min	simple path	NP-complete	D, Hoffmann 2001
PUSH- $*$ X	$\infty$	no	min	simple path	NP-complete	Hoffmann 2000

# Sokoban = Push-1FS

[Thinking Rabbit  
/ Hiroyuki  
Imabayashi 1984]



05 | moves: 0001 pushes: 0000 time: 0:00:04



## Sokoban

by *Spectrum Holobyte*™ a division of Sphere Inc.

Designed by Khaled Bentebal  
Programmed by Farah Soebrata  
Graphics by Jody Sather

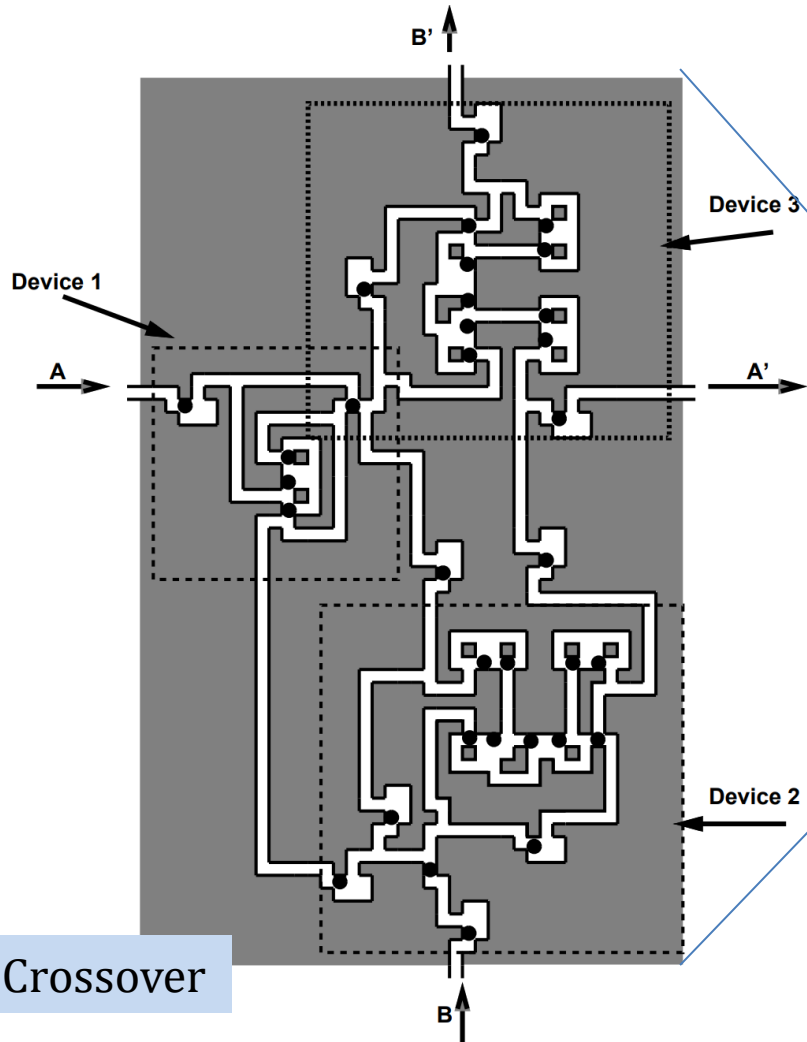
© Copyright 1984 ASCII Corp.



# Sokoban is PSPACE-complete [Culberson 1997]

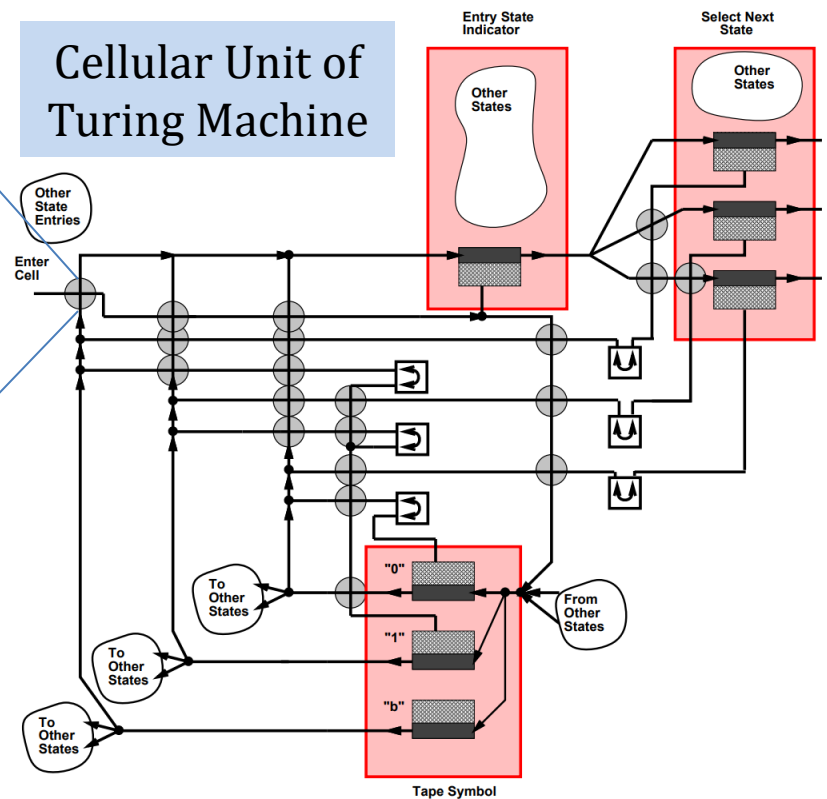


Chinook team, 1992



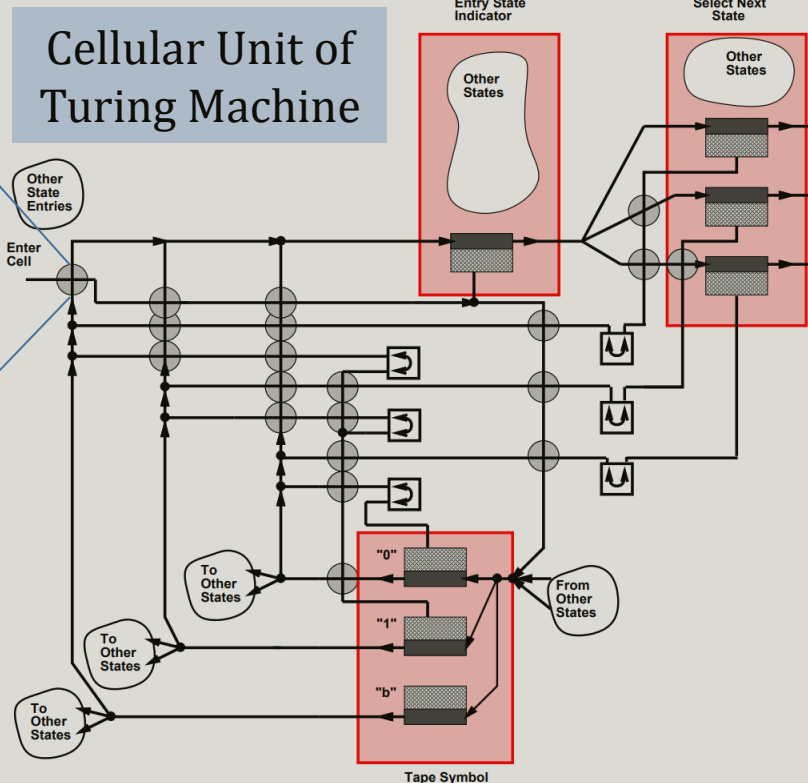
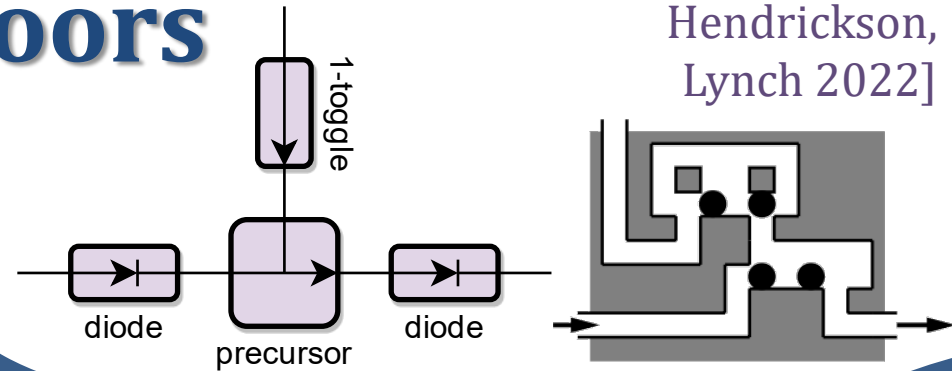
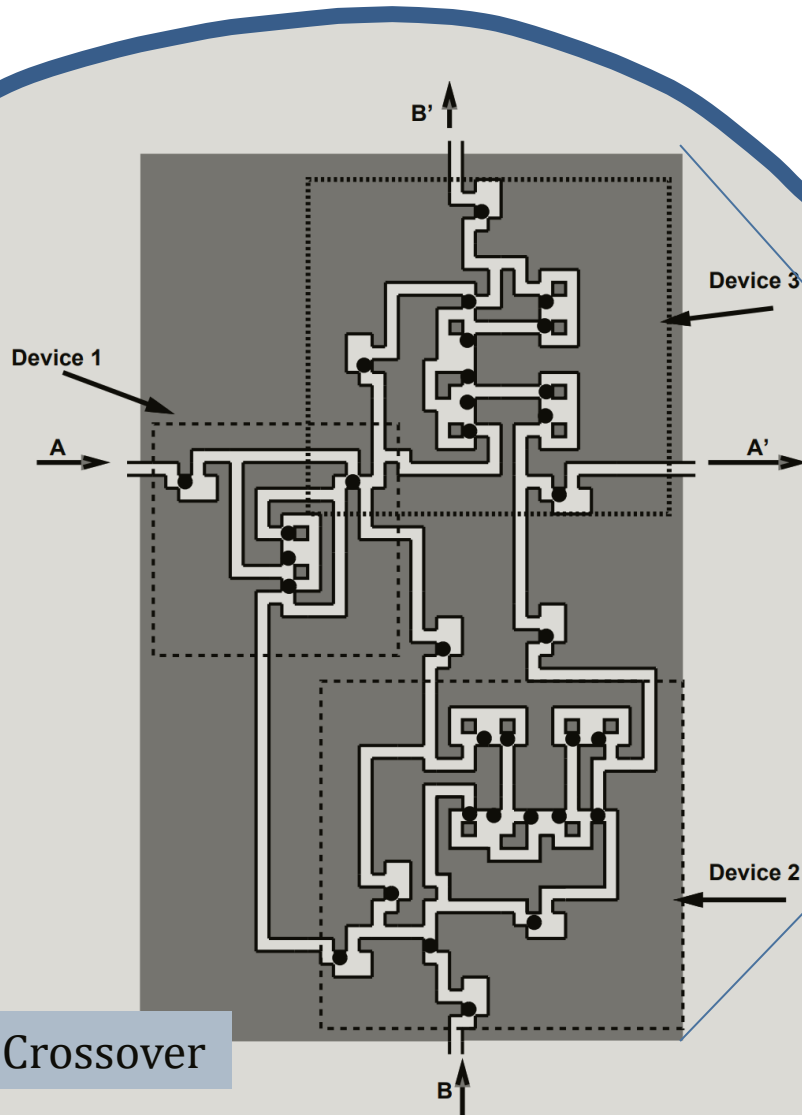
Crossover

## Cellular Unit of Turing Machine



# Sokoban is PSPACE-complete via Self-Closing Doors

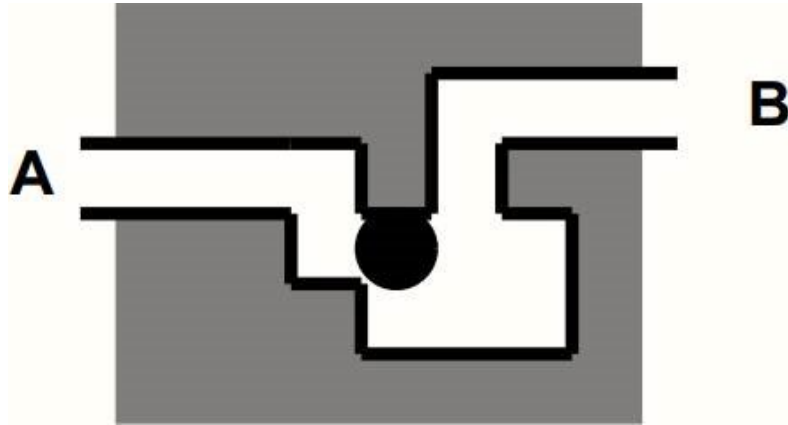
[Ani, Chung, Demaine, Diomidov, Hendrickson, Lynch 2022]





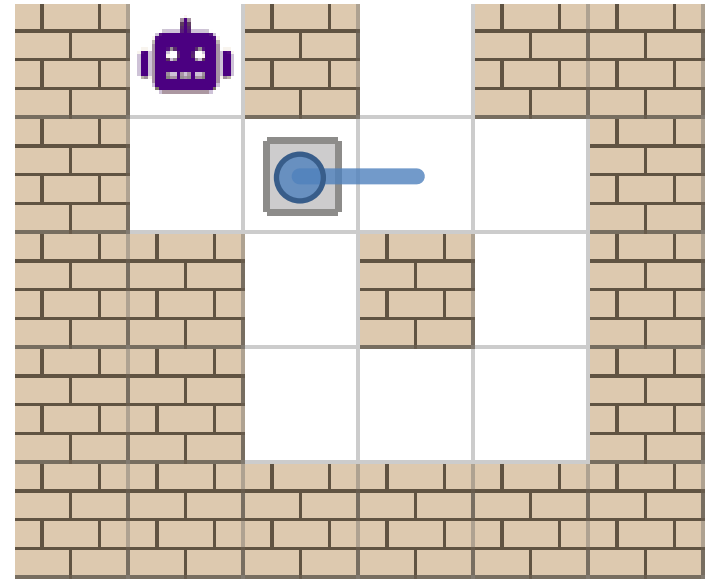
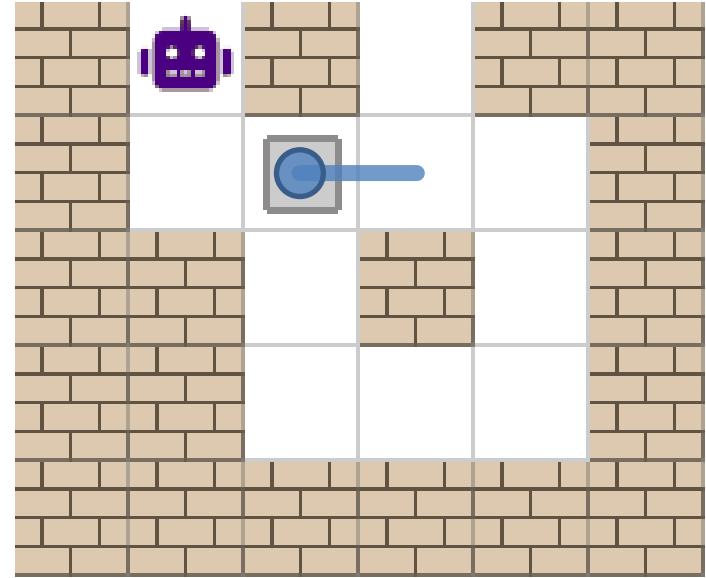
# Culberson's Diode Gadget [1998]

(*"one-way device"*)



intended use

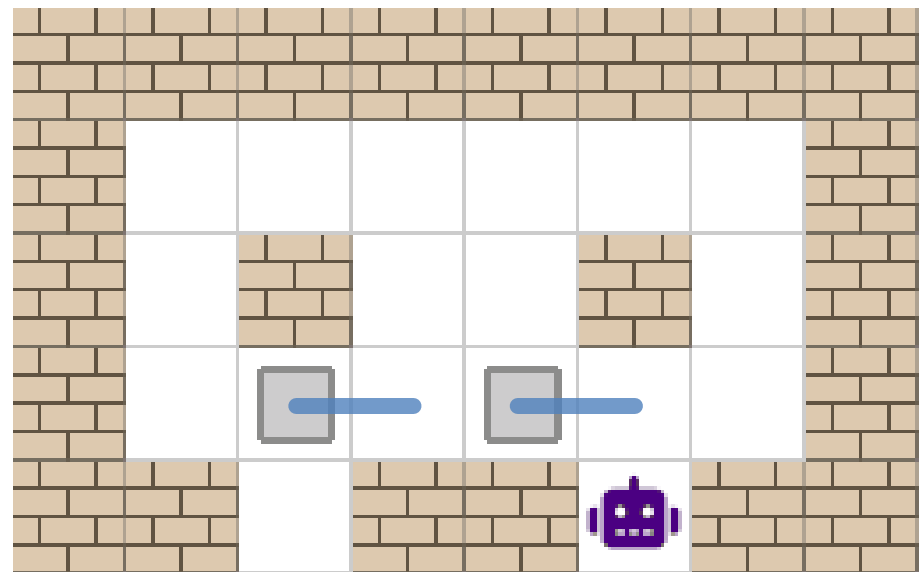
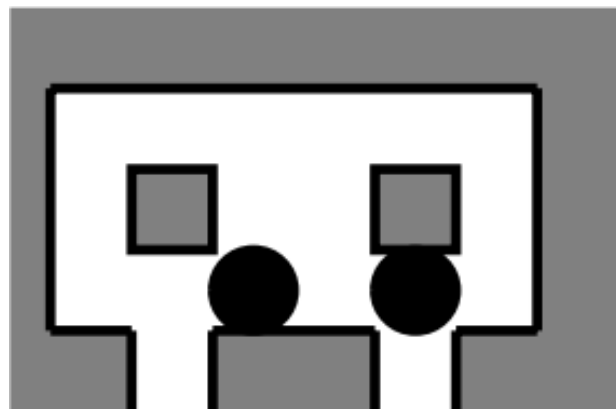
breaking use



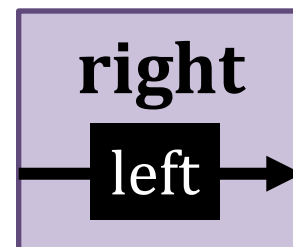
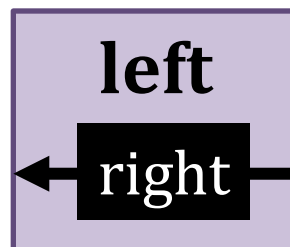


# Culberson's 1-Toggle Gadget [1998]

(*"reverser"*)



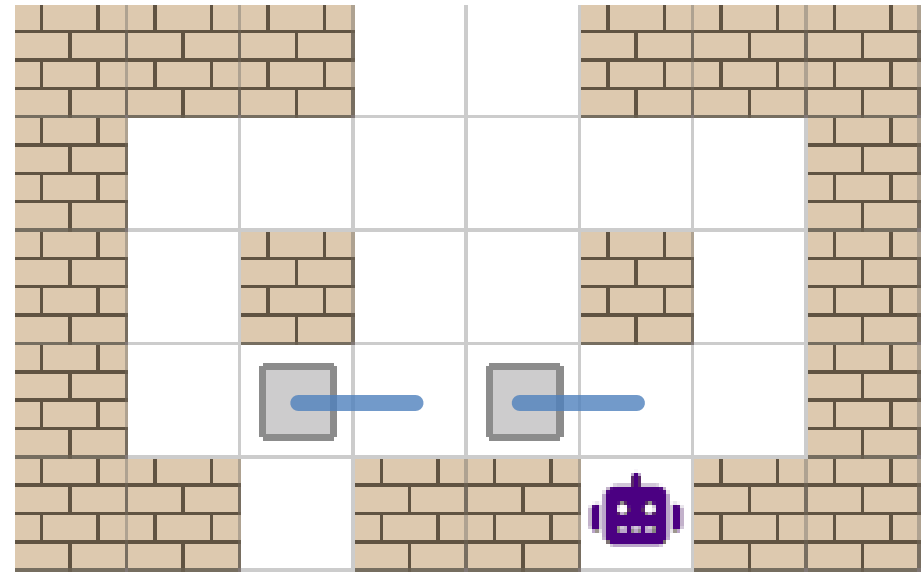
**1-Toggle**



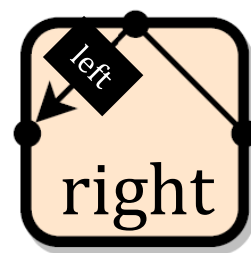
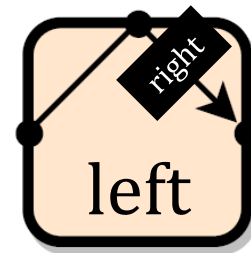
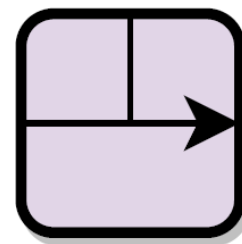


# Precursor Gadget

(implicit in Culberson 1998)



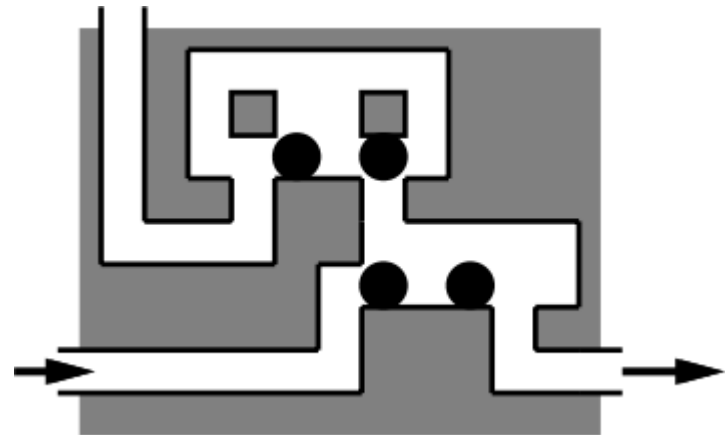
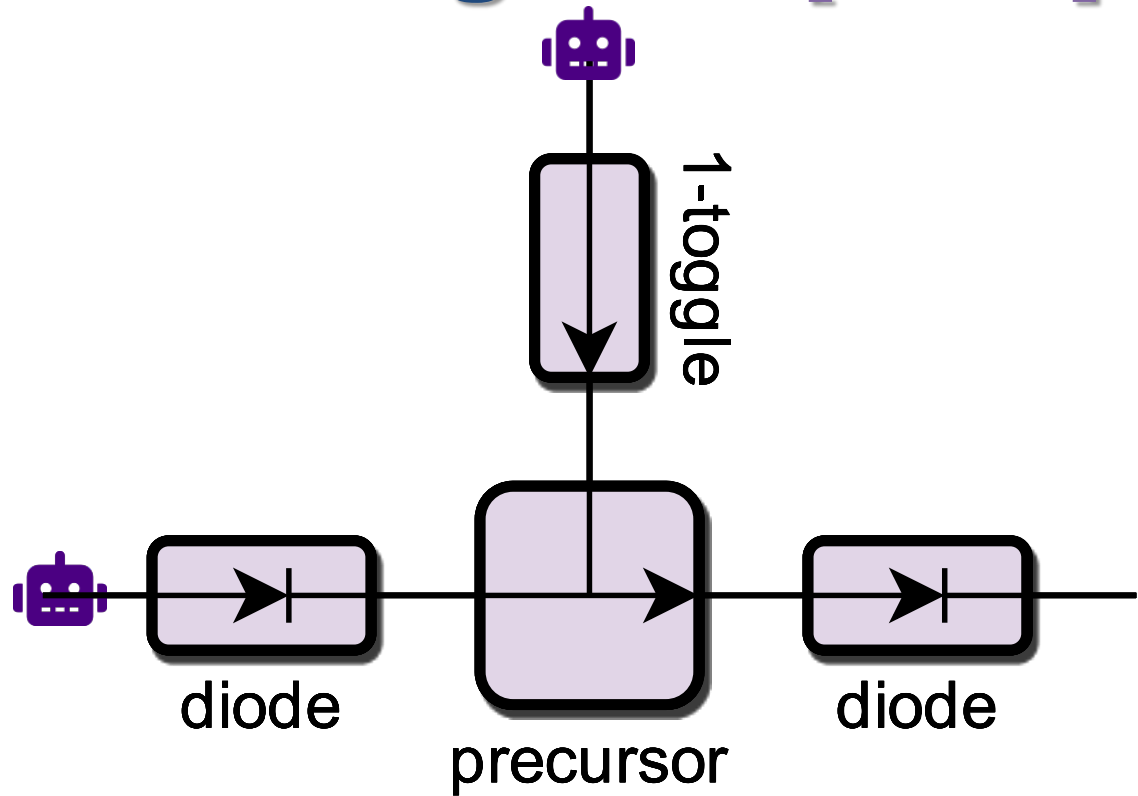
- Remembers last lower exit
- Cannot enter at other lower exit



# Culberson's Self-Closing Door [1998]

(*“pass-reset”*)

- **Self-close** = left to right
- **Open** = top to middle to top

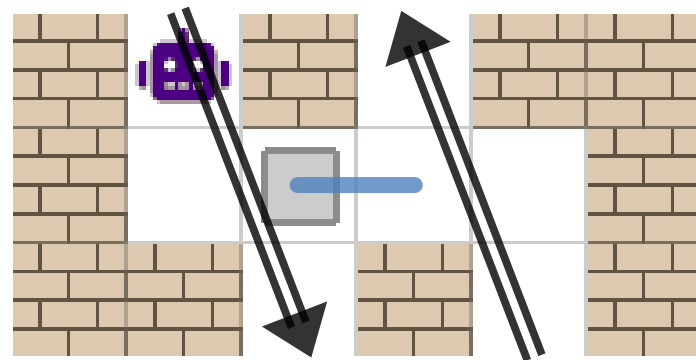
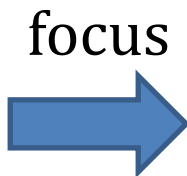
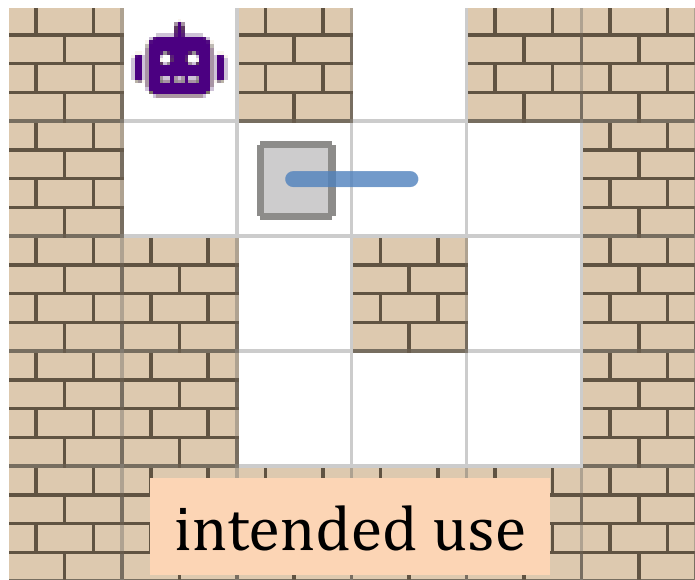




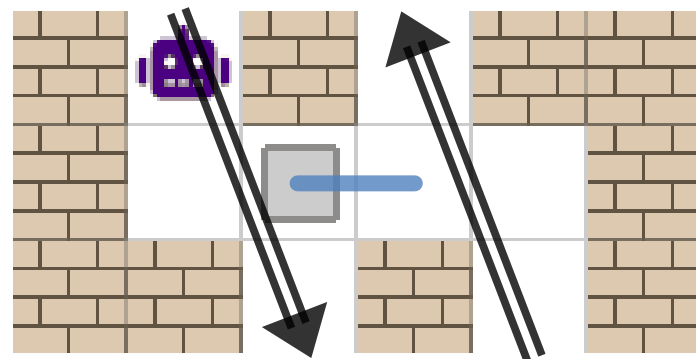
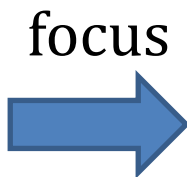
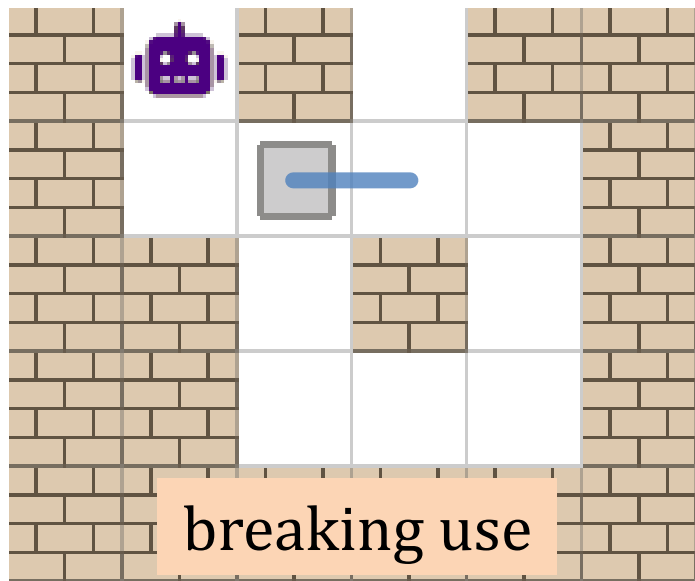


# Checkable Diode

[Ani, Chung, Demaine, Diomidov,  
Hendrickson, Lynch 2022]



**checking traversals**  
*possible*

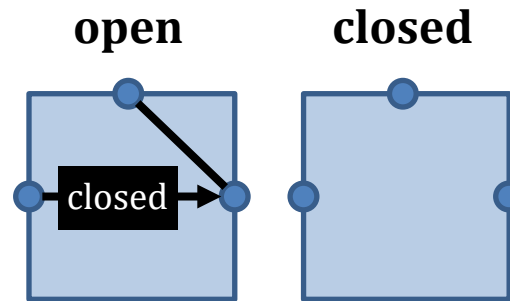


**checking traversals**  
*impossible*

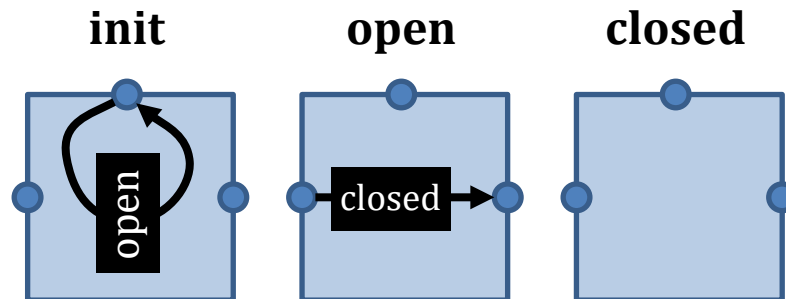
# Checkable Gadgets Framework

[Ani, Chung, Demaine,  
Diomidov, Hendrickson,  
Lynch 2022]

merged single-use  
closing gadget (MSC)



single-use opening  
gadget (SO)

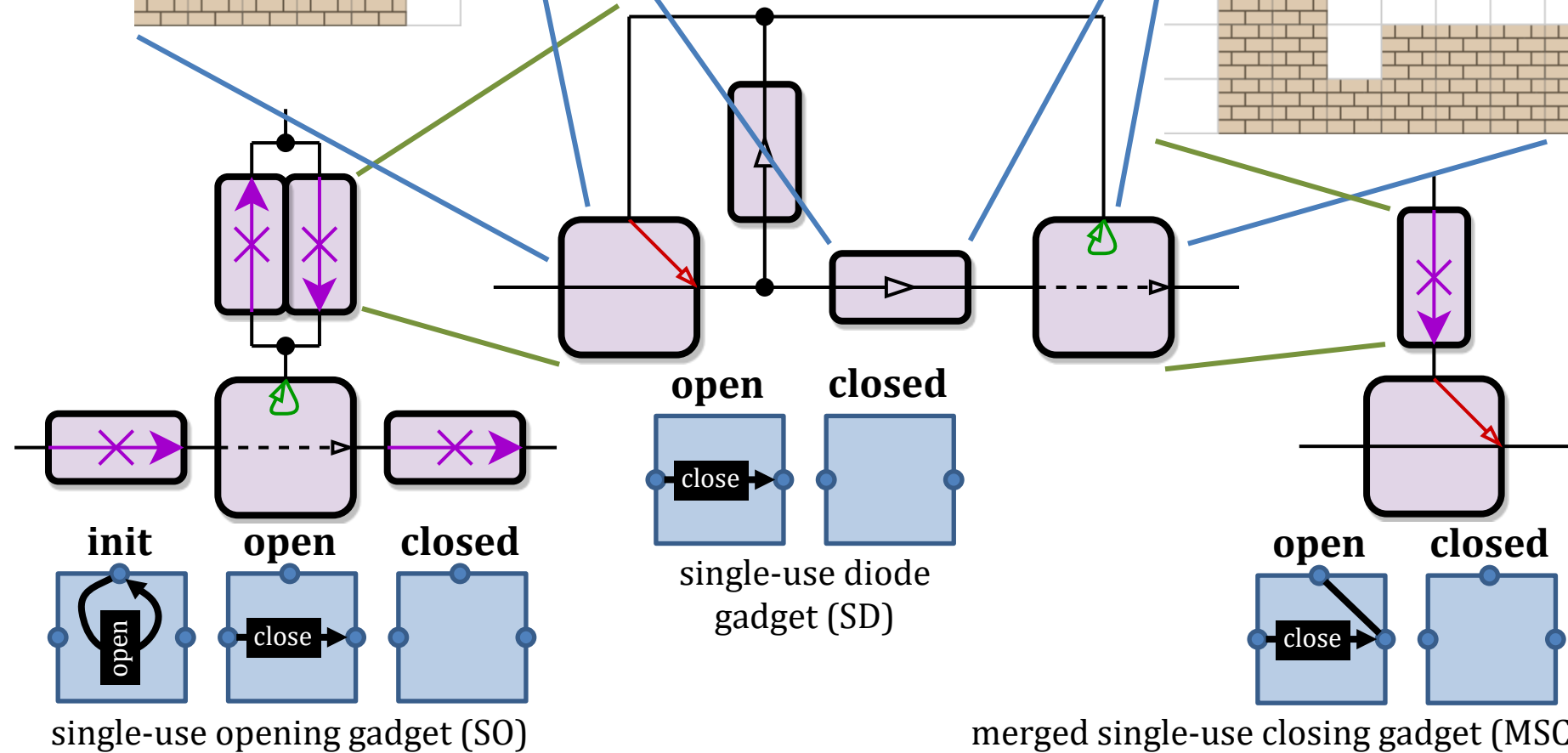
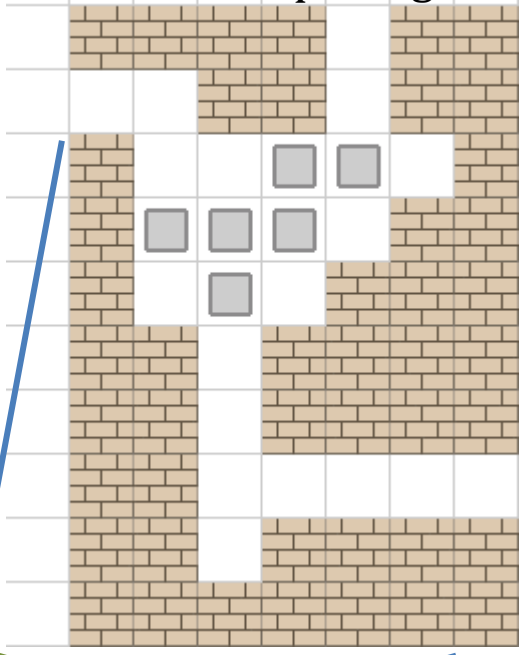
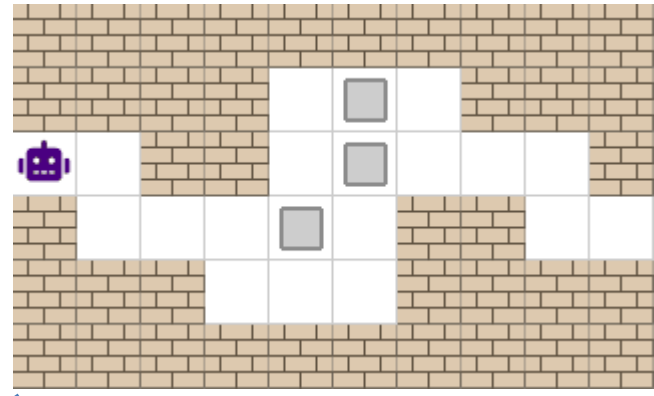
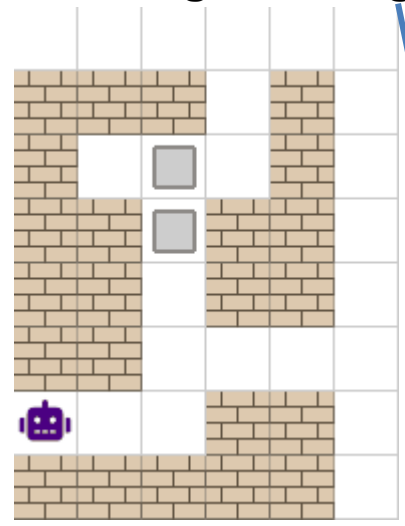




weak merged closing

no-return [DDO 2000]

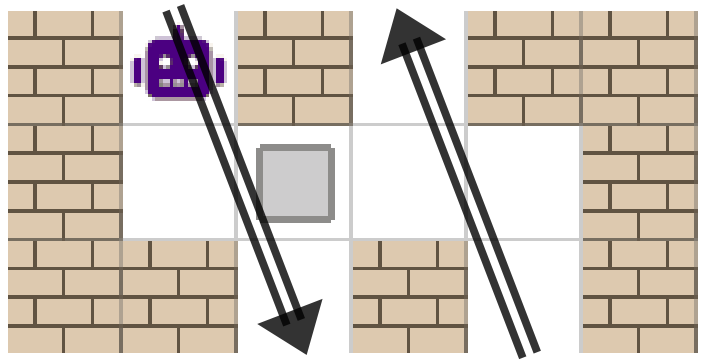
weak opening



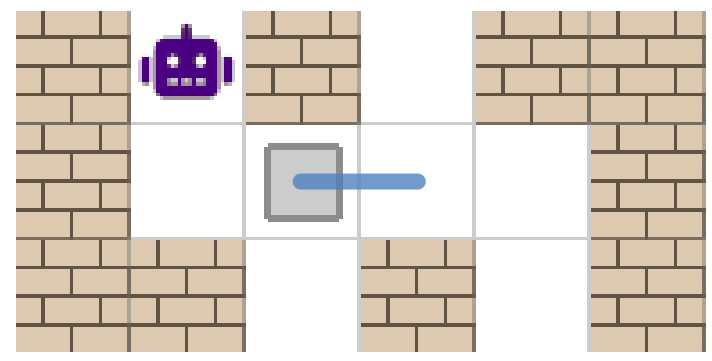
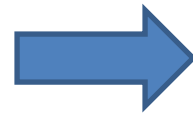


# Checkable Diode

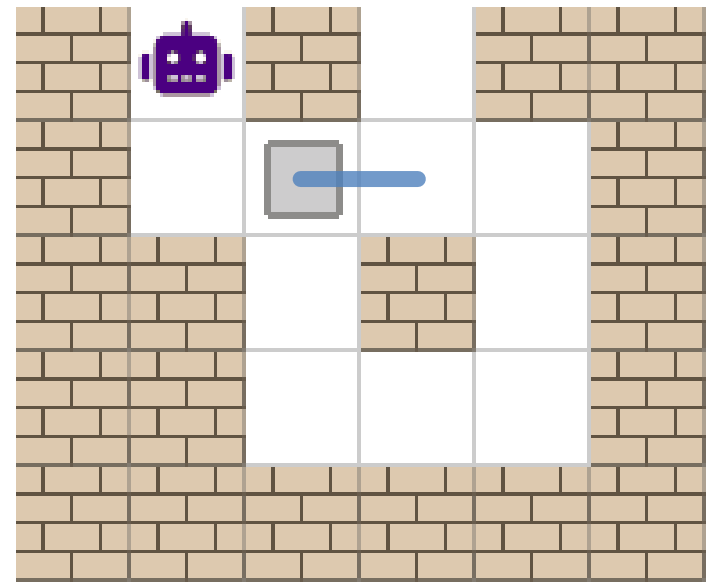
[Ani, Chung, Demaine,  
Diomidov, Hendrickson,  
Lynch 2022]



postselect



connect

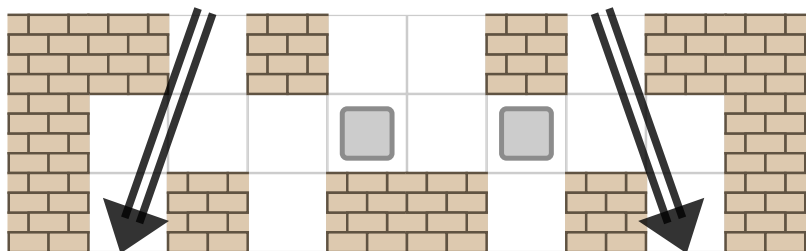


diode

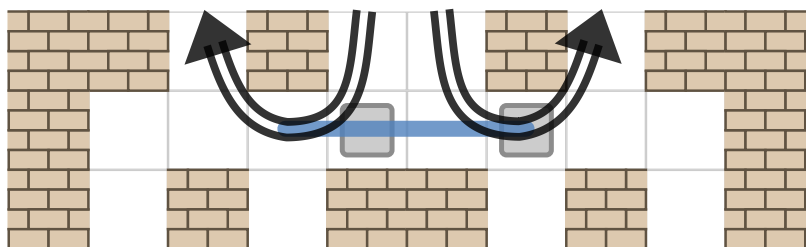
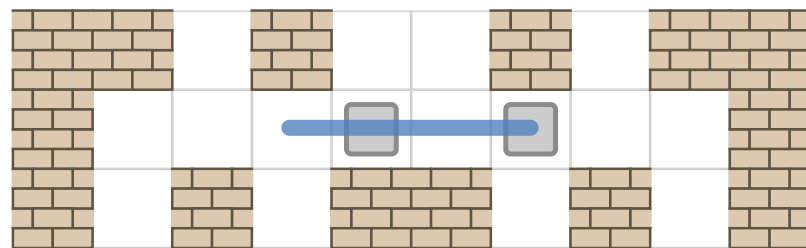
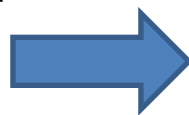


# Checkable Precursor

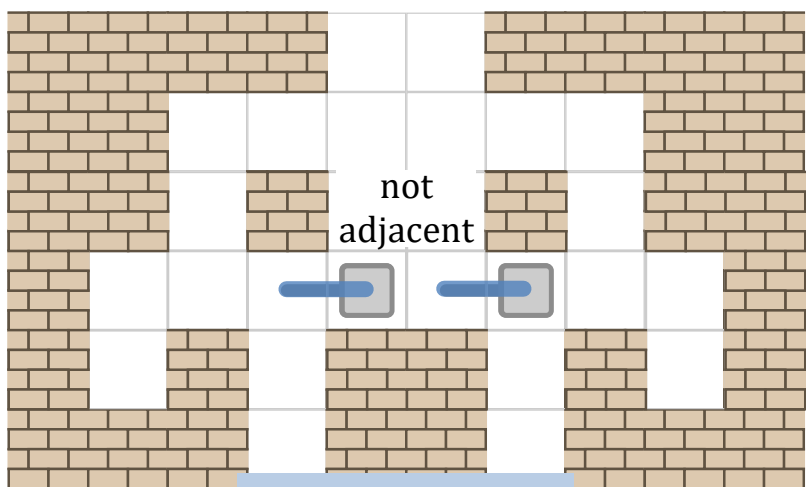
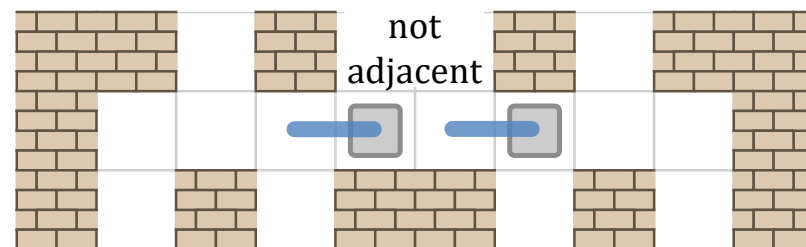
[Ani, Chung, Demaine,  
Diomidov, Hendrickson,  
Lynch 2022]



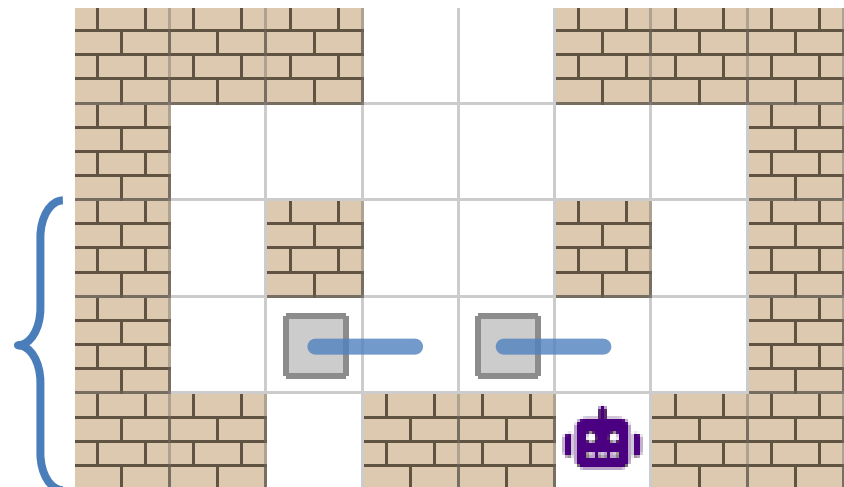
postselect



postselect



precursor

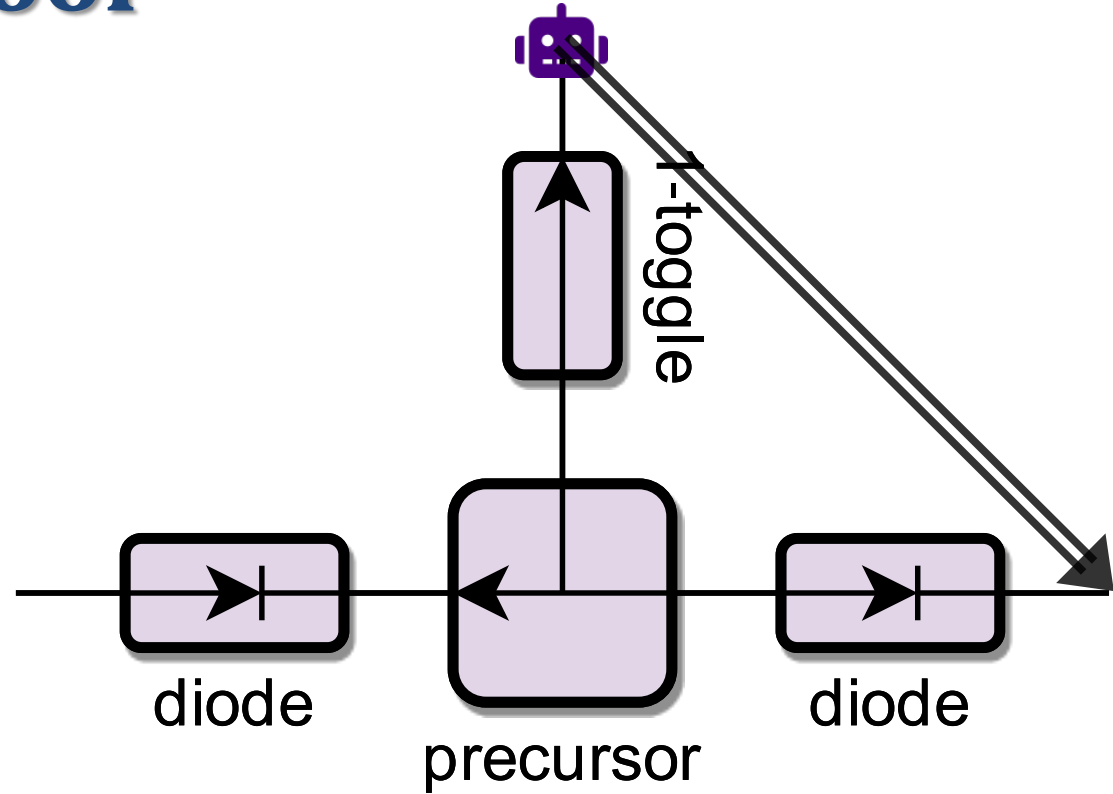




# Push-1F Self-Closing Door

[Ani, Chung, Demaine,  
Diomidov, Hendrickson,  
Lynch 2022]

- Postselect



# 2-State Deterministic Output-Disjoint Input/Output Gadgets

Output disjoint: no 2 inputs traverse to same output  
⇒ 2-state subunits:  $Q = \{\text{up}, \text{down}\}$

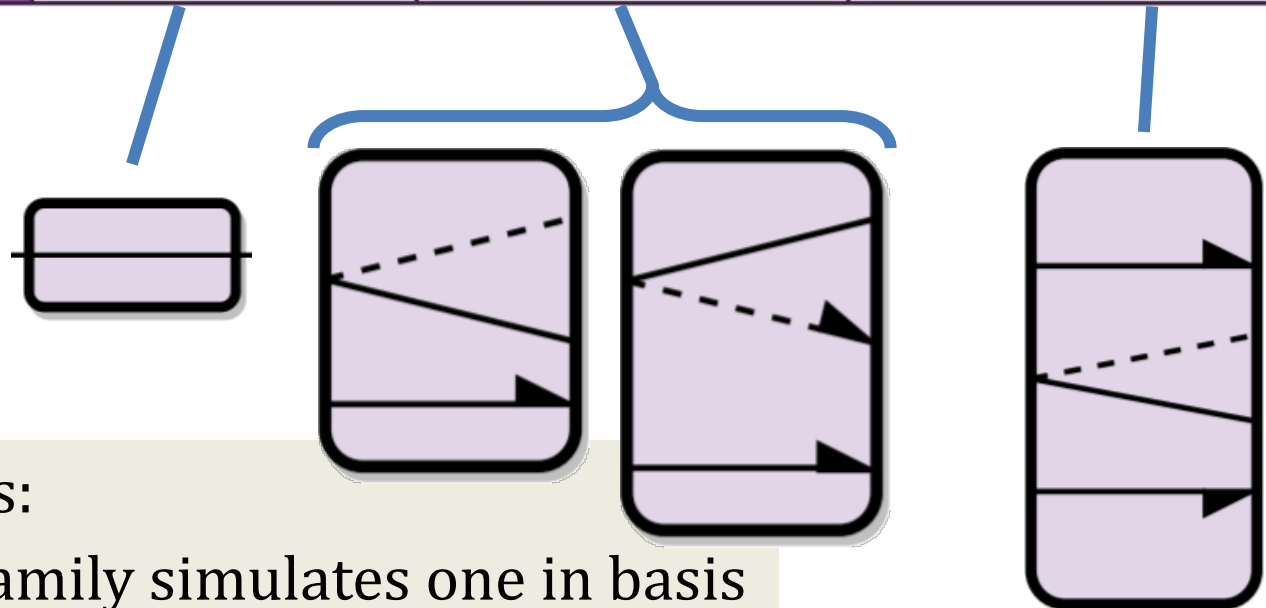
- switch: output depends on state
- set-up: sets state to up (sym. → down)
- toggle: flips state
- set-up switch: switch + set-up
- toggle switch: switch + toggle

unbounded

# Complexity of Multi-Input Input/Output Gadgets

[Ani, Demaine, Hendrickson, Lynch 2020]

	Trivial (No state change or on tunnels)	Bounded, multiple nontrivial inputs	Unbounded, multiple nontrivial inputs
0-Player (Fully Deterministic) [§2]	L	P-complete [§2.2]	PSPACE-complete [§2.3]
1-Player [§3]	NL-complete	NP-complete	PSPACE-complete



“Basis” of gadgets:  
 Every gadget in family simulates one in basis

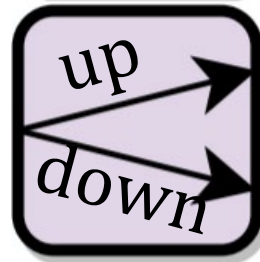


# Complexity of Single-Input Input/Output Gadgets

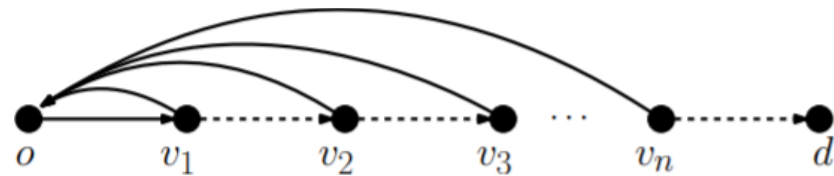
[Ani, Demaine, Hendrickson, Lynch 2020]

	Contained in	Hard for
0-Player (Fully Deterministic) [§2]	UP $\cap$ coUP [GHH <sup>+</sup> 18]	NL [FGMS18], [§2.1]
1-Player [§3]	NP [FGMS18], [§3.1]	NP [FGMS18], [§3.2]
2-Player [§4]	EXPTIME [FGMS18], [§4]	PSPACE [FGMS18], [§4]

- **ARRIVAL:**



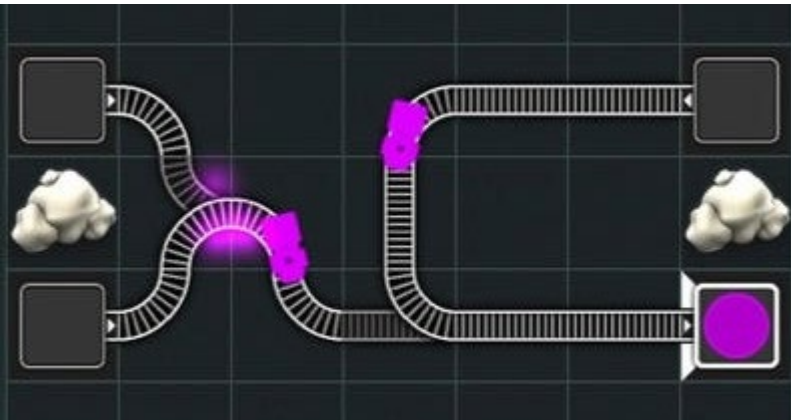
toggle switch



- $\in \text{NP} \cap \text{coNP}$  [Dohran, Gärtner, Kohler, Matoušek, Welzl 2017]
- $\in \text{UP} \cap \text{coUP}$  [Gärtner, Hansen, Hubacek, Kral, Mosaad, Slivova '18]
- $2^{\tilde{O}(\sqrt{n})}$  algorithm [Gärtner, Haslebacher, Hoang 2021]

# Applications to PSPACE-complete Sims

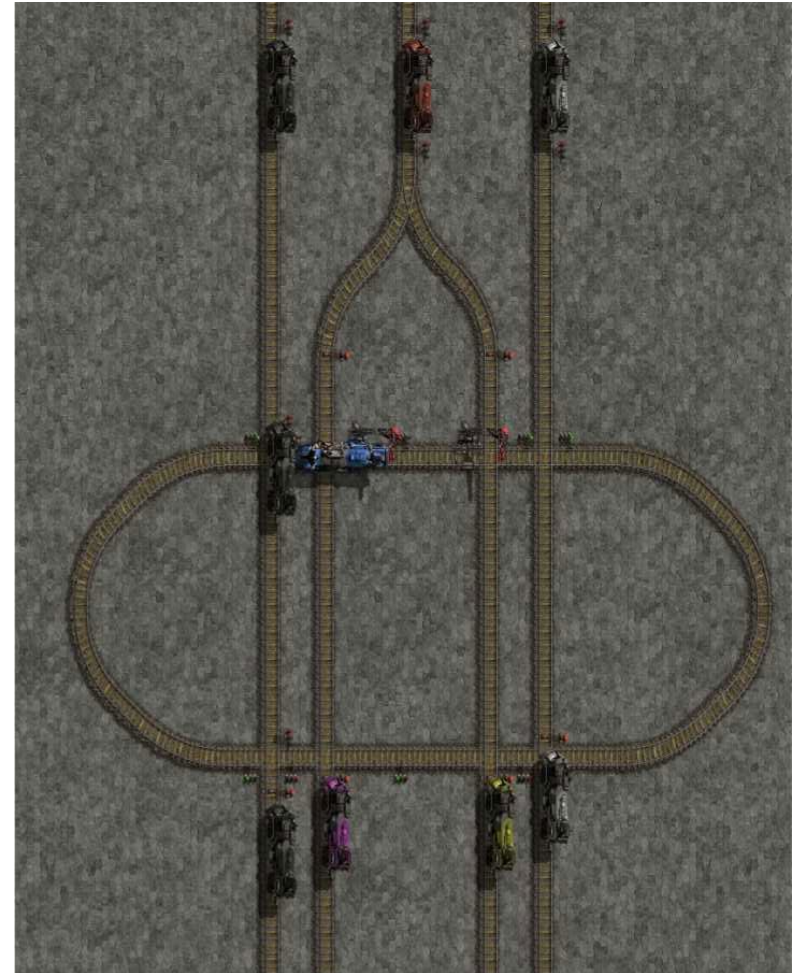
[Ani, Demaine, Hendrickson, Lynch 2020]



Trainyard with 1 train & color



[the Sequence]



Factorio trains or transport

# DAG Gadgets

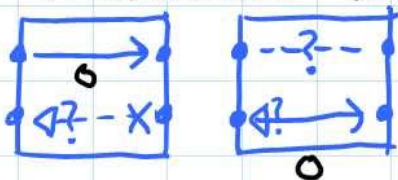
Hardness characterization: [Demaine, Hendrickson, Lynch 2000]

set of DAG tunnel gadgets have NP-complete reachability problem

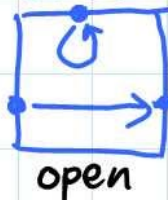
↳ not planar!

⇔ some gadget has distant opening: traversal that opens another tunnel

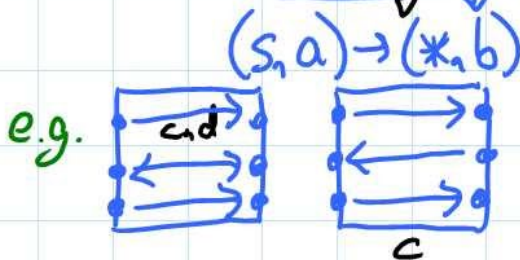
in some direction



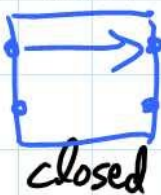
cf. opening door with 1 button [LG]



OR some gadget has forced distant closing: orientation of the tunnels + traversal  $(s, a \rightarrow b)$  that always closes  $\geq 1$  other tunnel in direction of orientation



cf. closing door [LG]



loops ← LDAG/ eventually static, not DAG

— otherwise, in P (in fact, NL)