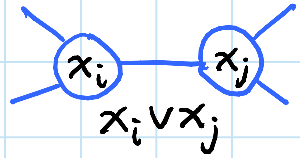
6.890	Lecture 9	Oct. 2, 2014

Vertex cover: [L7] [Karp 1972]

- choose k vertices to hit all edges in a graph
- $\equiv$ positive 2SAT with k true variables:



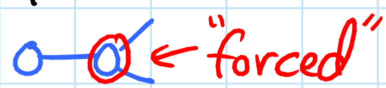
- NP-hard even for planar max-deg.-3 graphs
  [Garey & Johnson — SIJAM 1977] [L7] [Lichtenstein 1982]

Polynomial:

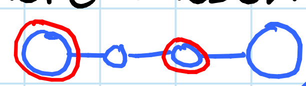
- Exact vertex cover: each edge hit exactly once
- Edge cover: choose edges to hit all vertices
  ($\approx$ matching)

Connected vertex cover: [Garey & Johnson — SIJAM 1977]

- vertex cover must induce connected graph
- NP-hard for planar max-deg.-4 graphs
- reduction from previous problem
  - never need to choose leaf $\Rightarrow$



  - add forced polygon in each face
  - increase each original vertex degree up to 4
  - subdivided edge needs exactly 1 more chosen (if otherwise a vertex cover)



    - never useful to choose both
  - added exactly 5 · # edges + to cover
  - these additions induce connected graph & every original vertex is adjacent

Rectilinear Steiner tree:

- given n points in the plane
- connect via horizontal & vertical segments of minimum total length ($\Rightarrow$ tree)

- reduction from previous problem:

  [Garey & Johnson – SIJAM 1977]

  - draw (max-deg.-4) graph rectilinearly on grid
  - scale by $4n^2$
  - points at all integer points along edges EXCEPT within radius 1 of vertices
  - connections may as well go through vertices
  - every edge must connect to a vertex $\to 2|E|$
  - must also connect other end of edges in a spanning tree of $G[VC]$ $\to 2(|V|-1)$

k-coloring: (vertex) AKA Chromatic Number

- given graph & positive integer k
- find color assignment $c: V \to \{1, 2, \ldots, k\}$ such that no edge $\{v, w\}$ is $\underbrace{\text{monochromatic}}_{c(v)=c(w)}$
- like $\underbrace{\text{XOR 2SAT}}_{x_i \neq x_j}$ but with k-valued logic
- NP-hard [Karp 1972]
- 2-coloring (bipartiteness) is polynomial
- 3-coloring NP-hard [Garey, Johnson, Stockmeyer —TCS 1976]
  - reduction from 3SAT:
    - colors gadget (3 distinct colors)
    - variable gadget: red & blue
    - clause gadget: red x_i's force red forward in Δ; else can put red back
  - reduction to planar 3-coloring:
    - crossover gadget: $x = x'$ & $y = y'$ (center alternates)
  - reduction to planar max. degree 4:
    - high-degree gadget: $x = x' = \cdots$ (Δs force copying)

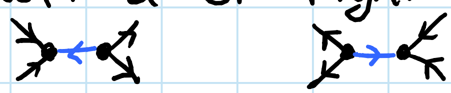
  - polynomial for max. degree 3: possible $\Leftrightarrow$ not K_4 [Brooks 1941]

Graph orientation: [Horiyama, Ito, Nakatsuka, Suzuki, Uehara – CCCG 2012]

- given an undirected 3-regular graph with 3 vertex types, find a valid orientation
  - 1-in-3: exactly 1 incoming edge, 2 outgoing
  - 2-in-3: exactly 2 incoming edges, 1 incoming
  - 0-or-3: exactly 0 or 3 incoming/outgoing edges
- NP-complete by reduction from 1-in-3SAT
- in plane, also need crossover gadget

Packing L trominoes: [Horiyama, Ito, Nakatsuka, Suzuki, Uehara]
& earlier by [Moore & Robson – DCG 2001]

- given grid polygon, can we pack k L's inside?
- exact packing: $k = \text{area}/3$
- NP-hard by reduction from Graph Orientation
  - "double 0-or-3" ≈ left 2 or right 2



Packing I trominoes: similar
& earlier by [Beaquier, Nivat, Remila, Robson – CGTA 1995]

Linear layout of graph = bijection $f: V \to \{1, 2, \dots, |V|\}$
- maps edges to segments in 1D

(see survey by Díaz, Petit, Serna 2002)

Bandwidth = minimize length of longest edge

Minimum linear arrangement = minimize total edge length
- e.g. motivated by VLSI layout

Cutwidth = minimize maximum # edges cut by a vertical line
- sum version ≡ min. linear arrangement

Vertex separation = minimize maximum (over x coords.) # vertices on left with edges on right

Sum cut = minimize sum of ↗

Edge bisection = minimize # edges crossing middle x

Vertex bisection = minimize # vertices in left half with edges to right half

Betweenness: given triples of the form "y is between x & z" [Opatrny — SICOMP 1979]

$x<y<z$ or $x>y>z$

find valid linear ordering

Bipartite crossing number: [Garey & Johnson – SIADM 1983]

minimum # crossings in bipartite graph drawing with 2 sides on 2 parallel lines

- reduction from Minimum Linear Arrangement
- $|E|^2$ edges between top_i & $bottom_i$
- $\Rightarrow$ top order = bottom order if $< |E|^4$ crossings
- edge $\{v_i, v_j\} \to$ edge $\{top_i, bottom_j\}$
- # crossings $= |E|^2 (k - |E| + 1) - 1$

OPT MLA (k) — -1 per edge ($-|E|$) — $\underbrace{+1) - 1}$ $< |E|^2$ for edge crossings

Crossing number: [Garey & Johnson – SIADM 1983]

draw graph with min. # crossings

- reduction from previous problem
- $3k+1$ copies of infrastructure
  - ↳ bipartite # crossings

Rubik's Cube: [Demaine, Demaine, Eisenstat, Lubiw, Winslow – ESA 2011]

- min # moves for $n \times n \times 1$ Rubik "square"
- $\Theta(n^2/\lg n)$ in worst case
- NP-hard with "don't cares" (missing stickers) by reduction from Betweenness
- $\boxed{\text{OPEN}}$: all stickers [Erickson 2010]