6.5440	Lecture 4	Sept. 16/18, 2014

The most important NP-complete (logic) problem family!

SAT = Satisfiability: [Cook 1971; Levin 1973]
- given a Boolean formula (AND, OR, NOT) over n variables $x_1, x_2, \ldots, x_n$
- can you set x_i's to make formula true?

Circuit SAT: formula expressed as acyclic circuit of gates (allows re-use)

CNF SAT: formula = AND of clauses [Cook 1971]
↓ Conjunctive Normal Form
clause = OR of literals
literal $\in \{x_i, \text{NOT } x_i\}$
- can view as bipartite graph: variables vs. clauses, positive/negative edges

3SAT: clause = OR of 3 literals [Cook 1971]
i.e. clause degrees = 3 (but allow repeats)

3SAT-3: each variable occurs in ≤3 clauses
- E3SAT-4 but E3SAT-3 ∈ P [Tovey-DAM 1984]
  ↳ exactly 3 distinct literals per clause

$\neg x_1 \vee x_2$
$\neg x_2 \vee x_3$
$\vdots$
$\neg x_k \vee x_1$

Monotone 3SAT: [Gold-I&C 1978] also -3
each clause all positive or all negative

Beware polynomial-time variants!

2SAT: clause = OR of 2 literals
- polynomial
- x OR $y \equiv$ NOT $x \Rightarrow y$ ($\equiv$ NOT $y \Rightarrow x$)
- guess x_i, follow all implication chains to check OK

BUT...

Max 2SAT: set variables to maximize # true clauses
- NP-complete [Garey, Johnson, Stockmeyer 1976]

Horn SAT: each clause has ≤ 1 positive literal
- NOT x OR NOT y OR NOT z OR w
- $\equiv$ NOT (x AND y AND z) w
- $\equiv$ (x AND y AND z) $\Rightarrow w$ [Horn 1951]
- $\Rightarrow$ polynomial like 2SAT

Dual-Horn SAT: each clause has ≤ 1 negative literal
↳ "weakly positive satisfiability" [Schaefer 1978]
- negate all variables → Horn SAT
- $\Rightarrow$ polynomial

Renamable Horn SAT: more generally, can negate some variables globally to make formula Horn
$\Rightarrow$ polynomial [Chandru, Coullard, Hammer, Montañez, Sun – Annals Math. & AI 1990]

DNF SAT: formula = OR of clauses
↓ Disjunctive Normal Form
clause = AND of literals
$\Rightarrow$ satisfiable $\Leftrightarrow \geq 1$ clause $\Rightarrow$ polynomial

Alternative clauses for 3SAT:

1-in-3SAT = exactly-1 3SAT [Schaefer 1978]
- clause = exactly 1 of 3 literals is true
  (⇒ 2 false ~ TFF, FTF, FFT)

↗ omitted by Schaefer

Positive 1-in-3SAT: no negations – all literals positive

BUT... ↗ sometimes called "monotone"

Positive not-exactly-1 3SAT: [Schaefer 1978]
- clause = 0, 2, or 3 variables are true
  i.e. $x_i \Rightarrow (x_j$ OR $x_k)$ → Dual Horn
- also require x_1 = TRUE (else set all x_i = FALSE)
  & x_2 = FALSE (or allow |clause| ≤ 3)
- polynomial

NAE 3SAT = not-all-equal 3SAT [Schaefer 1978]
- clause = 3 literals not all the same value
  (forbid FFF & TTT ⇒ 1 or 2 true, 2 or 1 false
  ~ whereas 3SAT forbids just FFF)
- nice symmetry between TRUE & FALSE

↗ omitted by Schaefer

Positive NAE 3SAT: no negations – all literals positive

Schaefer's Dichotomy: [Schaefer - STOC 1978]

- formula = AND of clauses
- general clause = relation on variables
  - assume in CNF (unique if minimal) ↳ use Karnaugh maps
  - ⇒ AND of subclauses

⇒ SAT is polynomial if either:

- setting all variables true or all variables false satisfies all relations
- subclauses are all Horn or all Dual Horn
- relations are all 2-CNF (subclause sizes ≤ 2)
- every relation can be expressed as a system of linear equations over $\mathbb{Z}_2$:

"XOR SAT" { $x_i \oplus x_j \oplus x_k \oplus x_\ell = 0$ or 1

↳ XOR ↳ Gaussian elimination

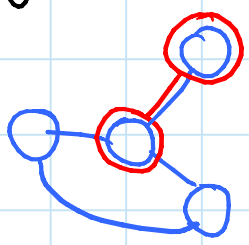
& otherwise, SAT is NP-complete!

Another hard version of SAT – seldom used?

2-colorable perfect matching: [Schaefer 1978]

- given a planar 3-regular graph
- 2-color the vertices such that every vertex has exactly 1 same-colored neighbor
- special case of 2-in-4SAT

(planarity & 3-regular left as exercise)



Pushing blocks:

- 1×1 robot navigating grid of blocks
- goal: get robot from start to target

- Push-k: robot can push up to k blocks at once
- Push-*: infinite strength
- PushPush: blocks slide until they hit something
- PushPushPush: blocks slide other blocks (ice) in chain reaction, up to strength k
- Push···-F: some blocks are fixed
- Push···-X: robot path cannot self-intersect (tiles disappear after traversal)

- Sokoban = Push-1F but with goal of filling target squares with blocks

Push-*: reduction from 3SAT [Hoffmann 2000]

- variable: push right in x_i or $\bar{x}_i$ row
  → fill in row of connection gadget
- connection: 1 free cell per occurrence of literal
- bridge: move up & block off leftward path
- clause: need a free spot below to traverse
- applies to PushPush-* & PushPushPush-* too

PushPush-1 in 3D: reduction from 3SAT
[O'Rourke & Smith Problem Solving Group 1999]

(Push)Push-1 (in 2D): reduction from 3SAT
[Demaine, Demaine, O'Rourke 2000]

- clause gadget, block other, lock gadget
- NAND crossover: not both N→S & W→E
- unidir. crossover: optional N→S, then W→E
  - can leak N→S then W→S, which can enable revisiting old variables; luckily fork gadget forces single T/F assignment

Phutball (Philosopher's Football) [Conway]

- one white stone (ball), many black stones (men)
- move = place new black stone
  OR "kick the ball" by jumping horizontally or vertically over black stones, immediately removing men, & repeat
- goal: reach opponent's side with ball

- PSPACE-hard [Dereniowski 2009]
- OPEN: EXPTIME-complete?
- NP-complete to decide mate-in-1:
  can you win in one move? (kick)
  - reduction from 3SAT [Demaine, Demaine, Eppstein 2000]

Checkers:

- EXPTIME-complete [Fraenkel, Garey, Johnson, Schaefer, Yesha 1978]
- mate-in-1 is polynomial
  - jumps preserve x & y parity
  ⇒ Euler path problem