

Recall: packing of n squares without rotation into a square is strongly NP-complete [L2]

Edge-unfolding polyhedra: given a polyhedron, cut along edges to unfold flat without overlap

- not always possible [Biedl et al. & Bern et al. 1998]
- strongly NP-hard [Abel & Demaine 2011]
- even for orthogonal polyhedra topologically sphere



Reduction from Square Packing:

- infrastructure: polyhedron with square with tower with squares & "atoms" on side
- "pipe" is super long but can move out
- ⇒ squares must pack inside base of tower
- atoms are universal: can turn left/right/straight in 2D unfolding & left/right/straight on tower surface
- ⇒ can connect & place squares as in any (slightly perturbed) packing, then exit via pipe
- lots of details e.g. shrink squares slightly to enable perturbation

Snake cube puzzle: AKA Cubra circa 1990

- given chain of unit cubes each with specified "turn angle" of 0 or 90° (elastic through centers)
 - goal: fold it into larger cube (exactly)
 - NP-hard
- [Abel, Demaine, Demaine, Eisenstat, Lynch, Schardl 2012]


Reduction from 3-Partition:

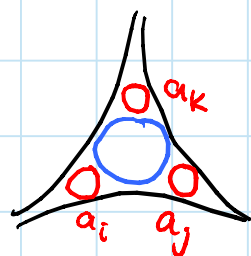
- infrastructure:
 - fill cube to leave $x \times y \times z$ box
 - fill box to leave "hub & slots" shape
 - each hub is $8 \times t \times \text{huge}$
- a_i gadget: $8a_i \downarrow$  must go in 1 hub
 - 8 to avoid coming back to same $4 \times 4 \times 4$ voxel
- connected together by zig-zag gadget 
- zig-zag is universal:
 - $2 \times 2 \times 2$ can turn/go straight
 - \Rightarrow fill Hamiltonian shapes scaled $2 \times$
 - $2 \times 2 \times 2$ refinement makes any shape Hamilt.
 - \Rightarrow $4 \times 4 \times 4$ refinement makes fillable by zig-zag
- parity issue: snake alternates in cell parity
- claim: can start & end at any faces of opposite parity

Disk packing: pack n given disks into given shape

- motivation: computational origami design
(tree method - see Lang)
- strongly NP-hard [Demaine, Fekete, Lang-OSME 2010]

Reduction from 3-Partition:

- infrastructure:
 - build $n/3$ symmetric  pockets
 - equilateral Δ : forced packing
 - square target: forced packing
 - + repeated subdivision with forced packings
 - + fill all other pockets by repeatedly adding maximal disks, until small enough (depth $\approx \lg n$)
- triple gadget: (in symmetric pocket)
 - scale a_i 's & t so that $t = 1$
 - shrink center disk by $-1/N$
 - shrink a_i disk by $-1/N a_i$, ^{↪ big}
grow it by $+a_i/N$
 - key property: disks fit $\Leftrightarrow a_i + a_j + a_k \leq t$
(proof by geometry + Taylor series)



Clickomania: [Schuessler ~2000?]

- given rectangular grid of colored squares
- move = remove connected group of >1 square of the same color
- remaining squares fall within each column
- empty columns disappear

- polynomial for one row or column $S \rightarrow \Delta | SS |$
- reduces to CFG parsing $c_i S c_i |$
 $c_i S c_i S c_i$

- NP-hard for

- 2 columns & 5 colors
- 5 columns & 3 colors
- OPEN: 2 rows?
2 colors?

[Biedl, Demaine,
Demaine, Fleischer,
Jacobsen, Munro 2000]

Reduction from 3-Partition: \rightarrow necessary: encoding in unary

- left column mostly checkerboard except middle & interspersed red \square s to measure t 's
 - collapses \Leftrightarrow red \square s removed
- right column has a_i groups \leftarrow scaled by $B = \frac{4}{3}n$
+ red squares on top
- details: spacing out groups & reds while still getting alignment

Tetris: [Alexey Pajhitnov 1985]

- rectangular board
- tetromino blocks come one at a time
↳ 4 unit squares joined edge-to-edge


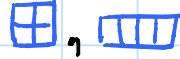




- can rotate block as it falls from sky
- filled lines disappear
- stack to sky \Rightarrow die


- perfect information version:

- know entire sequence of pieces to come
- initial board position given
- NP-complete to [Breukelaar, Demaine, Hohenberger, Hogeboom, Kusters, Liben-Nowell 2003]
 - survive
 - approximate # lines/Tetrises/time until death up to a factor of $n^{1-\epsilon}$

Reduction from 3-Partition: \rightarrow necessary: encoding in unary

- initial board = $n/3$ buckets of "depth" t
- a_i encoded as  $^{\alpha_i}$, 
- claim: entire gadget must go in one bucket
- finale =  $^{n/3}$,  $^{\frac{5}{4}t+4}$

Reachability: essentially all Tetris boards are reachable from initial state
[Hogeboom & Kusters - IJIGS 2005]

- OPEN:**
- initially empty board
 - $O(1)$ rows or columns
 - restricted piece sets (e.g. )
 - no last-minute slides
 - 2-player: PSPACE-complete?
 - online Tetris?

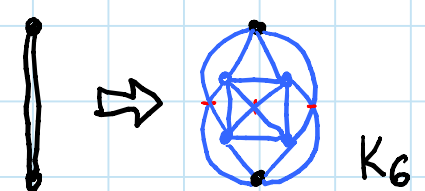
1-planarity: draw a given graph in the plane such that each edge crosses ≤ 1 other

[Ringel 1985]

- NP-complete [Grigoriev & Bodlaender - Alg. 2007]

Reduction from 3-Partition:

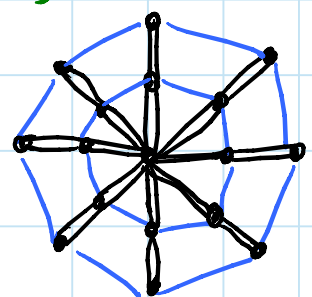
- uncrossable edge gadget:



(denoted by thick edge)

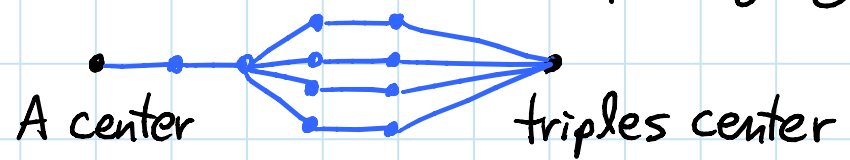
- double wheel gadget:

- unique embedding
- one for A
- one for triples



- separate triples with thick edges every t hours around triples gadget

- a_i gadget:



GeoLoop & Ivan's Hinge puzzles: piano-hinged dissection

↳ NP-complete from 3-Partition

Ruler folding:

- given carpenter's ruler with lengths a_1, a_2, \dots, a_n
- goal: fold to fit in 1D box of length L
- weakly NP-complete [Hopcroft, Joseph, Whitesides - ^{SICOMP} 1985]
- pseudopolynomial (like 2-Partition)

Reduction from (2-)Partition:

- idea: Partition solvable \Leftrightarrow can assign signs to a_i 's such that $\sum_i \pm a_i = 0$
- folding flips sign; unfolding leaves sign
- \Rightarrow can fold ends together \Leftrightarrow Partition solvable
- construction: $2B, B, a_1, a_2, \dots, a_n, B, 2B$
↳ $\sum a_i$
- \Rightarrow $2B$'s will be aligned & fit inside length- $2B$ box
 \Leftrightarrow can fold ends together \Leftrightarrow Partition solvable

Map folding (simple): given crease pattern, can it fold flat by sequence of simple folds?

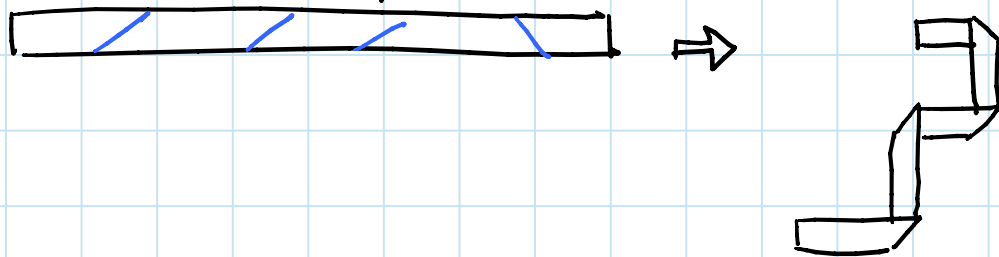
- weakly NP-hard [Arkin, Bender, Demaine, Demaine, Mitchell, Sethia, Skiena - 2000]

for orthogonal paper & orthogonal creases
or square paper & 45°/orthog. creases

Reduction from Partition:

- similar to Ruler Folding
- 2 vertical creases check y extent against frame
- horizontal creases done before or after check
if ruler folded ↙ ↘ if not

- force square paper into orthogonal shape:



OPEN: strongly NP-hard?
pseudopolynomial?

UPDATE: strongly NP-hard

[Akitaya, Demaine, Ku - JIP 2017]