6.5440	Class Week 3	Sept. 18, 2023

Schaefer's Dichotomy, polymorphism view:

- finite → domain D for variable values e.g. $\{0,1\}$
- clause type = relation $C \subseteq D^k$ on k variables ← not literals ~ no negation
  - closed/preserved under operation $f: D^m \to D$
    if $\underbrace{(x_1, \dots, x_k), (y_1, \dots, y_k), \dots}_{m \text{ assignments}} \in C$ satisfy clause
    $\Rightarrow \underbrace{(f(x_1, y_1, \dots), \dots, f(x_k, y_k, \dots))}_{\text{applying } f \text{ elementwise}} \in C$ also satisfies clause
  - "f is a polymorphism on C"

Binary dichotomy: (→ Schaefer's) $D = \{0,1\}$ [Jeavons — TCS 1998]

- SAT (CSP) with set Γ of clause types is $\in P$
  $\Leftrightarrow$ all $C \in \Gamma$ closed under same operation among:
  - 0 } constants — $\leftrightarrow$ all-false satisfies
  - 1 } $m=1$ — $\leftrightarrow$ all-true satisfies
  - AND } $m=2$ — $\leftrightarrow$ Horn (≤ 1 positive literal)
  - OR } — $\leftrightarrow$ dual-Horn (≤ 1 neg. literal)
  - majority } $m=3$ — $\leftrightarrow$ 2SAT
  - minority } — $\leftrightarrow$ linear equations mod 2
- otherwise, NP-complete

$\Rightarrow$ given truth table, easy to tell $\in P$ vs. NP-c.

- but given CNF or DNF formula, NP-hard to tell
  [Brunner, Chung, Demaine, Diomidova — 6.892 2019]

AND preserves Horn: say $x_1 \vee \neg x_2 \vee \neg x_3$
- given two satisfying assignments x, y
- if x or y set var. 2 or 3 to false then so does elementwise x AND $y \Rightarrow$ satisfied
- else $x_1 = y_1 = \text{true} = x_1$ AND $y_1 \Rightarrow$ satisfied

OR preserves dual-Horn: say $\neg x_1 \vee x_2 \vee x_3$
- if x or y set var. 2 or 3 to true then so does elementwise x OR $y \Rightarrow$ satisfied
- else $x_1 = y_1 = \text{false} = x_1$ OR $y_1 \Rightarrow$ satisfied

Majority preserves 2SAT: say $x_1 \vee x_2$
- given three satisfying assignments x, y, z
- each satisfies var. 1 or 2 ~ pick one

$\Rightarrow \geq 2$ assignments pick a common var. i to satisfy
$\Rightarrow$ majority also satisfies var. $i \Rightarrow$ satisfied
(works no matter which vars. are negated)
- perhaps related: majority expressable in 2SAT:

$$\text{majority}(x_i, y_i, z_i) = (x_i \vee y_i) \wedge (x_i \vee z_i) \wedge (y_i \vee z_i)$$

Minority preserves linear eq. mod 2: e.g. $x_1 \oplus \cdots \oplus x_k = c$ ↗XOR
- $\text{minority}(x_i, y_i, z_i) = x_i \oplus y_i \oplus z_i$

$\Rightarrow \text{minority}(x_1, y_1, z_1) \oplus \cdots \oplus \text{minority}(x_k, y_k, z_k)$
$= (x_1 \oplus y_1 \oplus z_1) \oplus \cdots \oplus (x_k \oplus y_k \oplus z_k)$ ↓ commutative + associative
$= (x_1 \oplus \cdots \oplus x_k) \oplus (y_1 \oplus \cdots \oplus y_k) \oplus (z_1 \oplus \cdots \oplus z_k)$
$= c \oplus c \oplus c = c$

Nonbinary dichotomy: [Zhuk – J.ACM 2020]

- SAT/CSP with Γ is $\in P \iff \Gamma$ preserved by some weak near-unanimity operation f

  $\forall x,y: f(y,x,\dots,x) = f(x,y,\dots,x) = f(x,x,\dots,y)$

- NP-complete otherwise

[Dvořák & Kupec – ICALP 2015]

Planar dichotomy: $D=\{0,1\}$ [Kazda, Kolmogorov, Rolínek – T.Alg. 2018]

- clause types now cyclicly ordered
- only one new polynomial case: all $C \in \Gamma$ both:
  - self-complementary: $(x_1,\dots,x_k) \in C \iff (\neg x_1,\dots,\neg x_k) \in C$
  - for any $x,y \in dC$: $\quad dC = \{(x_1 \oplus x_2, x_2 \oplus x_3, \dots, x_k \oplus x_1) \mid (x_1,x_2,\dots,x_k) \in C\}$

    for any i with $x_i \neq y_i$:

    there is a $j \neq i$ with $x_j \neq y_j$:

    x with i & j flipped $\in dC$

  (dC is an "even Δ-matroid)

Example: Positive $NAE = \{(0,0,1), (0,1,1), \text{ \& shifts}\}$

$dNAE = \{(0,1,1), \cancel{(1,0,1)}, \text{ \& shifts}\}$

- $x=(0,1,1)$, $y=(1,1,0)$: if we flip one differing var. (e.g. 1), flip other (e.g. 3) to get y

Symmetric dichotomy: C symmetric in all k vars.
⇒ S-in-Ek SAT where $S \subseteq \{0, 1, \ldots, k\}$
- allow negated literals
- polynomial for:
  - $S = \emptyset$ — never sat.
  - $S = \{0, 1, \ldots, k\}$ — always sat.
  - $S = \{0\}$
  - $S = \{k\}$ } forces all literals (0 or 1)
  - $S = \{0, k\}$ — all equal
  - $S = \{0, 1\}$ — ≤ 1-in-k SAT
  - $S = \{k-1, k\}$ — $\ge (k-1)$-in k SAT
  - $S = \{$all odd ints. in $[0, k]\}$ — XOR SAT
  - $S = \{$all even ints. in $[0, k]\}$ — XNOR SAT

  } linear eqs. mod 2
- otherwise NP-complete
- roughly in [Brakensiek & Guruswami - SICOMP 2021] interpreted by [Alcock, Asif, Bosboom, Demaine, Filho, Hesterberg, Lynch, Urschel - 6.892 2019]

  see also [Ivan Tadeu Ferreira Antunes Filho - MEng 2019]