Inverse kinematics for 3R manipulator

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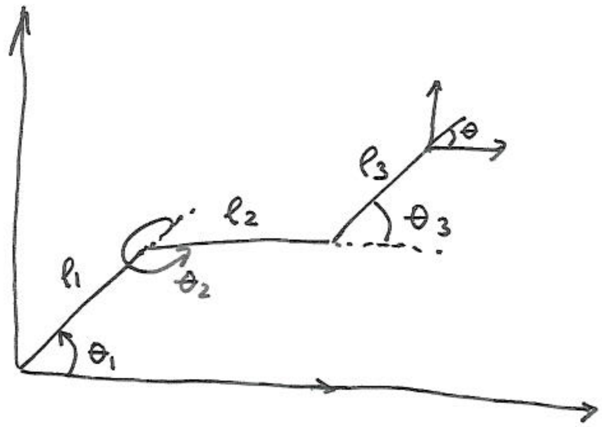

reference point (x, y, θ)

$$x = \ell_1 \cos\theta_1 + \ell_2 \cos(\theta_2 + \theta_1) + \ell_3 \cos(\theta_1 + \theta_2 + \theta_3)$$

$$y = \ell_1 \sin\theta_1 + \ell_2 \sin(\theta_1 + \theta_2) + \ell_3 \sin(\theta_1 + \theta_2 + \theta_3)$$

Given x, y, θ solve for $\theta_1, \theta_2, \theta_3$

$$\theta = \theta_1 + \theta_2 + \theta_3 \Rightarrow \begin{cases} x - \ell_3 \cos\theta = \ell_1 \cos\theta_1 + \ell_2 \cos(\theta_1 + \theta_2) \\ y - \ell_3 \sin\theta = \ell_1 \sin\theta_1 + \ell_2 \sin(\theta_1 + \theta_2) \end{cases}$$

2 equations in unknowns θ_1, θ_2

$$\begin{cases} x - \ell_3 \cos\theta = x' \\ y - \ell_3 \sin\theta = y' \end{cases}$$

(this is a renaming step since $x, y, \ell_3, \cos\theta, \sin\theta$ are known)

$$\Rightarrow \begin{cases} x' - \ell_1 \cos\theta_1 = \ell_2 \cos(\theta_1 + \theta_2) \\ y' - \ell_1 \sin\theta_1 = \ell_2 \sin(\theta_1 + \theta_2) \end{cases}$$

Now let's square both sides:

$$\begin{cases}(x'-l_1\cos\theta_1)^2=\left(l_2\cos(\theta_1+\theta_2)\right)^2 \\ (y'-l_1\,\mathrm{sim}\,\theta_1)^2=\left(l_2\,\mathrm{sim}\,(\theta_1+\theta_2)\right)^2\end{cases}$$ add them $\Rightarrow$

$$(x'-l_1\cos\theta_1)^2+(y'-l_1\,\mathrm{sim}\,\theta_1)^2=\left(l_2\cos(\theta_1+\theta_2)\right)^2+\left(l_2\,\mathrm{sim}\,(\theta_1+\theta_2)\right)$$

$$\Rightarrow x'^2-2x'l_1\cos\theta_1+l_1^2\cos^2\theta_1+y'^2-2y'l_1\,\mathrm{sim}\,\theta_1+l_1^2\,\mathrm{sim}^2\theta_1$$

$$=l_2^2\cos^2(\theta_1+\theta_2)+l_2^2\,\mathrm{sim}^2(\theta_1+\theta_2)=l_2^2\underbrace{\left(\cos^2(\theta_1+\theta_2)+\mathrm{sim}^2(\theta_1+\theta_3)\right)}_{=1}=$$

$$\Rightarrow(-2x'l_2\cos\theta_1)+(-2y_1l_1)\,\mathrm{sim}\,\theta_1+(x'^2+y'^2+l_1^2-l_2^2)=$$

$\Rightarrow$ This is an equation in one unknown θ_1 of the form

$$P\cos\alpha+Q\,\mathrm{sim}\,\alpha+R=0$$

Let $t=\tan\frac{\alpha}{2}$; then

$$\cos\alpha=\frac{1-t^2}{1+t^2}\quad\text{and}\quad\sin\alpha=\frac{2t}{1+t^2},\ \text{so}$$

$$P\,\frac{1-t^2}{1+t^2}+Q\,\frac{2t}{1+t^2}+R=0$$

$$P(1-t^2)+2tQ=R(1+t^2)=0$$

$$P-Pt^2+2Qt+R+Rt^2=0$$

$$(R-P)t^2+2Qt+P+R=0$$

$$t_{1,2}=\frac{-2Q\pm\sqrt{4Q^2-4(P+R)(R-P)}}{2(R-P)}$$

$$=\frac{-Q\pm\sqrt{Q^2+R^2-P^2}}{R-P}$$

so $\alpha=2\,\mathrm{atan}\,t=\theta_1$; note this has 2 sol

To get θ_2 substitute in $x'-l_2\cos\theta_1=l_2\cos(\theta_1+\theta$

To get θ_3 substitute in $\theta_1+\theta_2+\theta_3=\theta$