Introduction to Algorithms May 7, 2004
Massachusetts Institute of Technology 6.046J/18.410J
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Problem Set 8 (Optional)

This problem set is **not due** and is meant as practice for the final. *Reading:* 26.1, 26.3, 35.1

Problem 8-1. Prove these problems are NP-Complete:

(a) SET-COVER: Given a finite set $\mathcal{U}$, a collection $\mathcal{S} = \{S_1, S_2, \ldots S_m\}$ of subsets of $\mathcal{U}$, and an integer k, determine whether there is a sub-collection of $\mathcal{S}$ with cardinality k that covers $\mathcal{U}$. In other words, determine whether there exists $\mathcal{S}' \subset \mathcal{S}$ such that $|\mathcal{S}'| = k$ and $\bigcup_{S_i \in \mathcal{S}'} S_i = \mathcal{U}$.

(b) DIRECTED-HAMILTONIAN-PATH: given a directed graph $G = (V, E)$ and two distinct vertices $u, v \in V$, determine whether G contains a path that starts at u, ends at v, and visits every vertex of the graph exactly once. (Hint: Reduce from HAM-CYCLE: 34.5.3 in CLRS.)

Problem 8-2. MAX-CUT Approximation

A *cut* $(S, V - S)$ of an undirected graph $G = (V, E)$ is a partition of V into two disjoint subsets S and $V - S$. We say that an edge $(u, v) \in E$ *crosses* the cut $(S, V - S)$ if one of its endpoints is in S and the other is in $V - S$. The MAX-CUT problem is the problem of finding a cut of an undirected connected graph $G = (V, E)$ that maximizes the number of edges crossing the cut. Give a deterministic approximation algorithm for this problem with a ratio bound of 2. *Hint:* Your algorithm should guarantee that the number of edges crossing the cut is at least half of the total number of edges.

Problem 8-3. Global Edge Connectivity of Undirected and Directed Graphs

(a) The global edge connectivity of an *undirected* graph is the minimum number of edges that must be removed to disconnect the graph. Show how the edge connectivity of an undirected graph $G = (V, E)$ can be determined by running the maximum-flow algorithm $|V| - 1$ times, each on a flow network with $O(|V|)$ vertices and $O(|E|)$ edges.

(b) The global edge connectivity of a *directed* graph G is the minimum number of directed edges that must be removed from G so that the resulting graph is no longer strongly connected. Show how the edge connectivity of a directed graph $G = (V, E)$ can be determined by running the maximum-flow algorithm $|V|$ times, each on a flow network with $O(|V|)$ vertices and $O(|E|)$ edges.

Problem 8-4. Perfect Matching in Regular Bipartite Graph

A bipartite graph $G = (V, E)$, where $V = L \cup R$, is *d-regular* if every vertex $v \in V$ has degree exactly d.

(a) Prove that for every d-regular bipartite graph, $|L| = |R|$.

(b) Model the maximum d-regular bipartite matching as a max-flow problem as in Section 26.3 in CLRS. Show that the max-flow value from s to t in the formulation is $|L|$.

(c) Prove that every d-regular bipartite graph has a matching of cardinality $|L|$.