P vs NP (Episode II)

• We defined a large class of interesting problems, namely NP
  – Decision problems (YES or NO)
  – Solvable in non-deterministic polynomial time.
    I.e., a solution can be verified in polynomial time

• We have a way of saying that one problem is not harder than another ($\Pi' \leq \Pi$)

• Our goal: show equivalence between hard problems
Reductions: $\Pi'$ to $\Pi$

$X' \xrightarrow{f} f(X') = X \xrightarrow{A \text{ for } \Pi} \text{YES} \xrightarrow{\text{YES}} \text{NO} \xrightarrow{\text{NO}} \text{A' for } \Pi'$
Showing equivalence between difficult problems

- Options:
  - Show reductions between all pairs of problems
  - Reduce the number of reductions (!) using transitivity of “≤”
  - Show that all problems in NP are reducible to a fixed $\Pi$. To show that some problem $\Pi' \in \text{NP}$ is equivalent to all difficult problems, we only show $\Pi \leq \Pi'$. 

Diagram:
- TSP
- Clique
- P3
- P4
- P5

$\Pi' \rightarrow \Pi$
The first problem \[\prod\]

- Satisfiability problem (SAT):
  - Given: a formula \(\varphi\) with \(m\) clauses \(C_1, \ldots, C_m\) over \(n\) variables.
  - Example: \(x_1 \lor x_2 \lor x_5, x_3 \lor \neg x_5\)
  - Check if there exists TRUE/FALSE assignments to the variables that makes the formula satisfiable
SAT is NP-complete

• Fact: SAT ∈ NP
• Theorem [Cook’71]: For any \( \Pi' \in \text{NP} \), we have \( \Pi' \leq \text{SAT} \).
• Definition: A problem \( \Pi \) such that for any \( \Pi' \in \text{NP} \) we have \( \Pi' \leq \Pi \), is called \text{NP-hard}
• Definition: An NP-hard problem that belongs to NP is called \text{NP-complete}
• Corollary: SAT is NP-complete.
Menu for today

SAT
↓
Clique
↓
Independent set
↓
Vertex cover

Conclusion: all of the above problems are NP-complete

(Thanks, Steve J)
Follow from Cook’s Theorem
Clique again

• Clique:
  – Input: undirected graph \( G=(V,E), K \)
  – Output: is there a subset \( C \) of \( V \), \(|C| \geq K\), such that every pair of vertices in \( C \) has an edge between them
SAT ≤ Clique

• Given a SAT formula \( \varphi = C_1, \ldots, C_m \) over \( x_1, \ldots, x_n \), we need to produce \( G = (V, E) \) and \( K \), such that \( \varphi \) satisfiable iff \( G \) has a clique of size \( \geq K \).

• Notation: a literal is either \( x_i \) or \( \neg x_i \)
SAT ≤ Clique reduction

- For each literal $t$ occurring in $\varphi$, create a vertex $v_t$
- Create an edge $v_t - v_{t'}$ iff:
  - $t$ and $t'$ are not in the same clause, and
  - $t$ is not the negation of $t'$
SAT ≤ Clique example

- **Formula:** \( x_1 \lor x_2 \lor x_3, \neg x_2 \lor \neg x_3, \neg x_1 \lor x_2 \)
- **Graph:**

- **Claim:** \( \varphi \) satisfiable iff \( G \) has a clique of size \( \geq m \)
Proof

• “$\rightarrow$” part:
  – Take any assignment that satisfies $\varphi$.
    E.g., $x_1=\text{F}$, $x_2=\text{T}$, $x_3=\text{F}$
  – Let the set $C$ contain one satisfied literal per clause
  – $C$ is a clique
Proof

• “$\leftarrow$” part:
  – Take any clique $C$ of size $\geq m$ (i.e., $= m$)
  – Create a set of equations that satisfies selected literals.
    E.g., $x_3 = T$, $x_2 = F$, $x_1 = F$
  – The set of equations is consistent and the solution satisfies $\varphi$
Altogether

- We constructed a reduction that maps:
  - YES inputs to SAT to YES inputs to Clique
  - NO inputs to SAT to NO inputs to Clique
- The reduction works in poly time
- Therefore, SAT ≤ Clique → Clique NP-hard
- Clique is in NP → Clique is NP-complete
Independent set (IS)

- **Input**: undirected graph $G=(V,E)$
- **Output**: is there a subset $S$ of $V$, $|S| \geq K$ such that no pair of vertices in $S$ has an edge between them
Clique ≤ IS

• Given an input \( G = (V,E), K \) to Clique, need to construct an input \( G' = (V',E') \), \( K' \) to IS, such that \( G \) has clique of size \( \geq K \) iff \( G' \) has IS of size \( \geq K \).

• Construction: \( K' = K, V' = V, E' = \overline{E} \)

• Reason: \( C \) is a clique in \( G \) iff it is an IS in \( G' \)'s complement.
Vertex cover (VC)

- **Input:** undirected graph $G=(V,E)$
- **Output:** is there a subset $C$ of $V$, $|C| \leq K$, such that each edge in $E$ is incident to at least one vertex in $C$. 
**IS \leq VC**

- Given an input $G=(V,E), K$ to IS, need to construct an input $G'=(V',E'), K'$ to VC, such that $G$ has an IS of size $\geq K$ iff $G'$ has VC of size $\leq K'$.

- Construction: $V'=V$, $E'=E$, $K'=|V|-K$

- Reason: $S$ is an IS in $G$ iff $V-S$ is a VC in $G$. 

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*Introduction to Algorithms*

May 8, 2003  L20.18