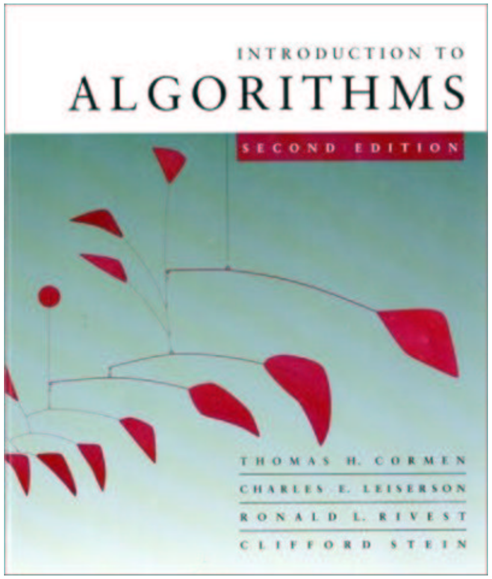
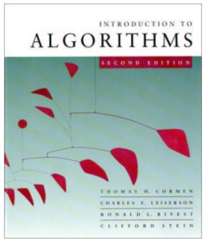
Introduction to Algorithms

6.046J/18.401



Lecture 20

Prof. Piotr Indyk



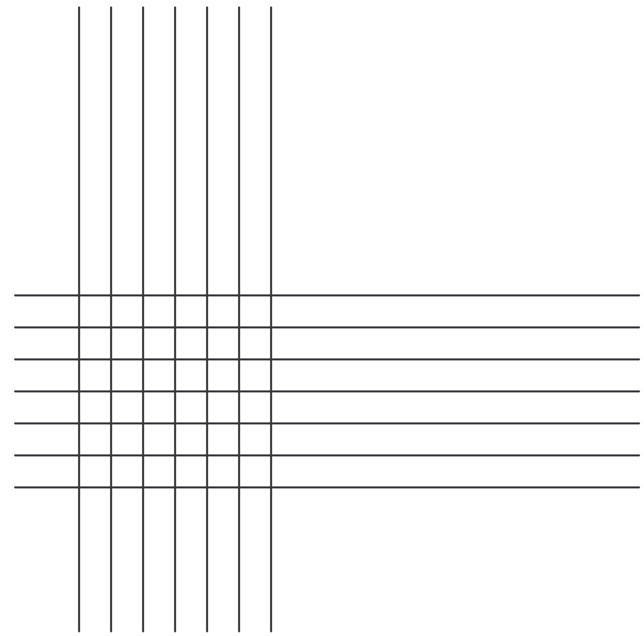
Computational Geometry ctd.

- Segment intersection problem:
  - Given: a set of n distinct segments $s_1 \ldots s_n$, represented by coordinates of endpoints
  - Goal (I): detect if there is any pair $s_i \neq s_j$ that intersects
  - Goal (II): report all pairs of intersecting segments

Segment intersection

- Easy to solve in $O(n^2)$ time
- …which is optimal for the reporting problem:
- However:
  – We will see we can do better for the detection problem
  – Moreover, the number of intersections P is usually small.

  Then, we would like an *output sensitive* algorithm, whose running time is low if P is small.

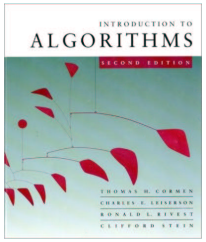


Result

- We will show:
  - $O(n \log n)$ time for detection
  - $O((n +P) \log n)$ time for reporting
- We will use …

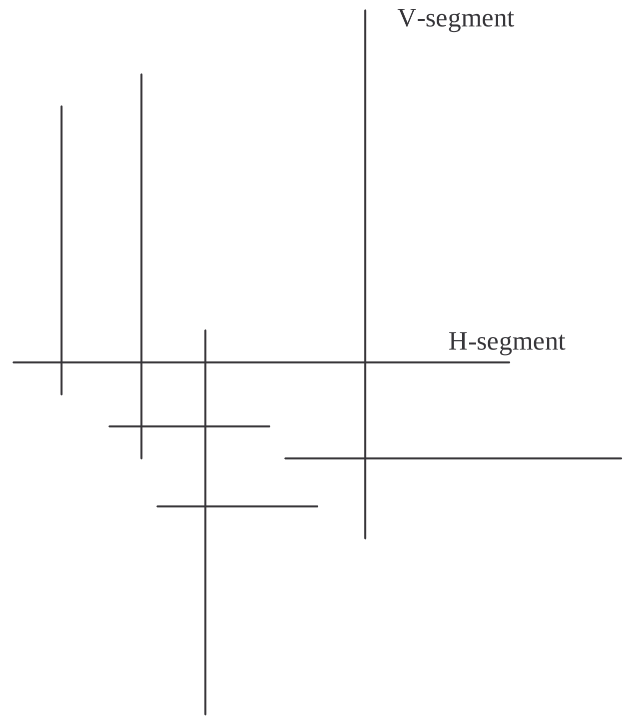
  … (no, not divide and conquer)

  … Binary Search Trees
- Specifically: *Line sweep approach*



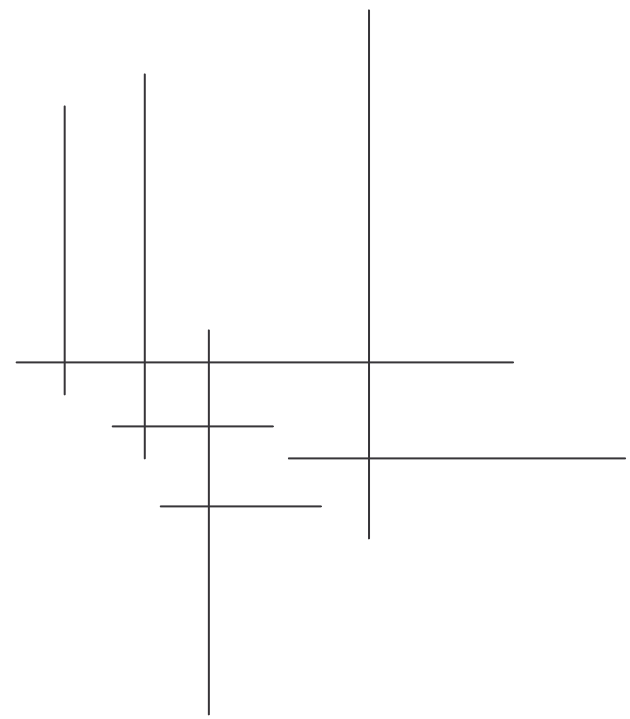
Orthogonal segments

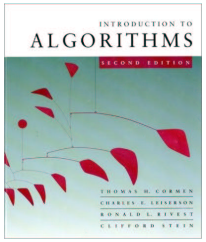
- All segments are either horizontal or vertical
- Assumption: all coordinates are distinct
- Therefore, only vertical-horizontal intersections exist



Orthogonal segments

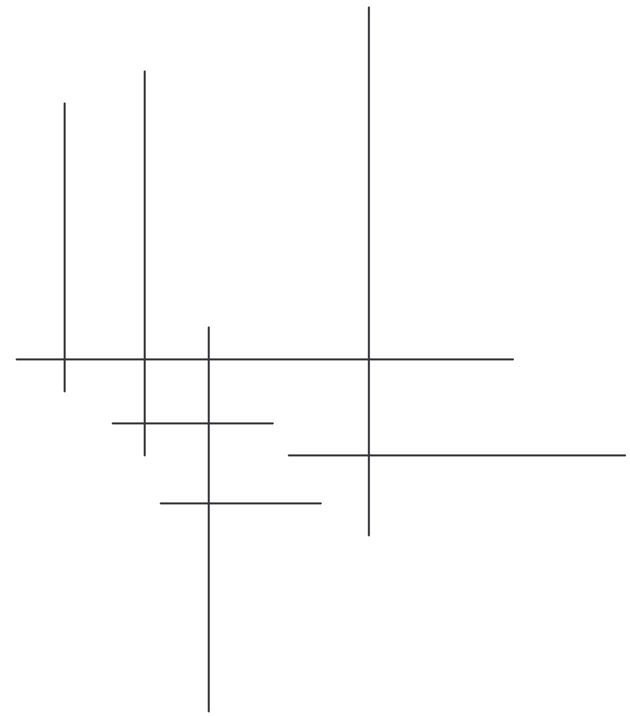
- Sweep line:
  - A *vertical line* sweeps the plane from left to right
  - It "stops" at all "important" x-coordinates, i.e., when it hits a V-segment or endpoints of an H-segment
  - Invariant: all intersections on the left side of the sweep line have been already reported





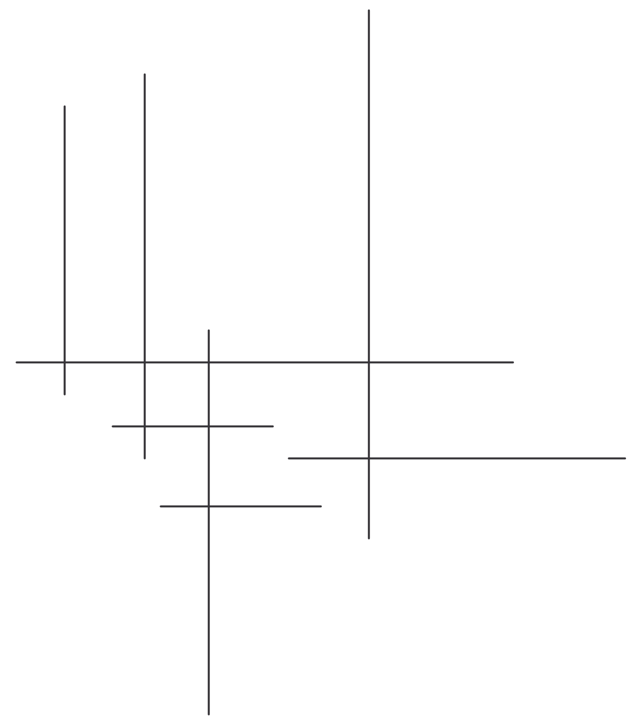
Orthogonal segments ctd.

- We maintain sorted y-coordinates of H-segments currently intersected by the sweep line (using a balanced BST T)
- When we hit the left point of an H-segment, we add its y-coordinate to T
- When we hit the right point of an H-segment, we delete its y-coordinate from T



Orthogonal segments ctd.

- Whenever we hit a V-segment (with coordinates y_{top}, y_{bottom}), we report all H-segments in T with y-coordinates in $[y_{top}, y_{bottom}]$



Algorithm

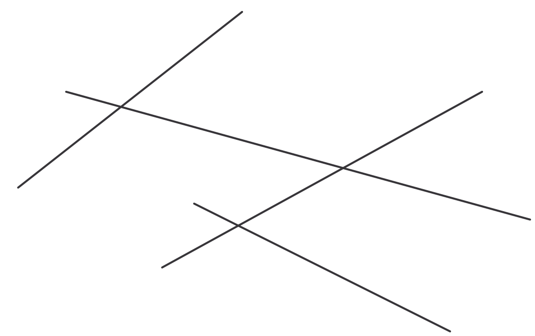
- Sort all V-segments and endpoints of H-segments by their x-coordinates – this gives the “trajectory” of the sweep line
- Scan the elements in the sorted list:
  - Left endpoint: add segment to T
  - Right endpoint: remove segment from T
  - V-segment: report intersections with the H-segments stored in T

Analysis

- Sorting: O(n log n)
- Add to/delete from T:
  - O(log n) per operation
  - O(n log n) total
- Processing V-segments:
  - O(log n) per intersection
  - O(P log n) total
  - Can be improved to O(P +n log n)
- Overall: O(P+ n log n) time

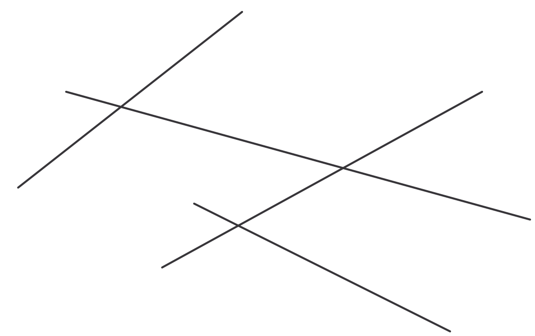
The general case

- Assumption: all coordinates of endpoints and intersections distinct
- In particular:
  - No vertical segments
  - No three segments intersect at one point



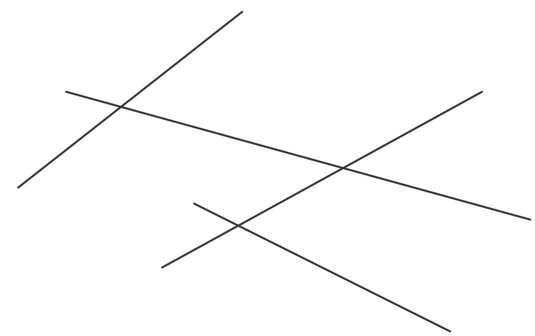
Sweep line

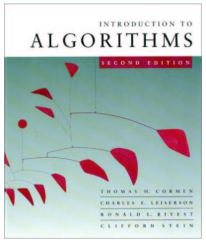
- Invariant (as before): all intersections on the left of the sweep line have been already reported
- Stops at all "important" x-coordinates, i.e., when it hits endpoints or intersections
- Do not know the intersections in advance !
- The list of important x-coordinates is constructed and maintained *dynamically*



Sweep line

- Also need to maintain the information about the segments intersecting the sweep line
- Cannot keep the values of y-coordinates of the segments !
- Instead, we will maintain their *order* .I.e., at any point, we maintain all segments intersecting the sweep line, sorted by the y-coordinates of the intersections.





Algorithm

- Initialize the "vertical" BST V (to "empty")
- Initialize the "horizontal" priority queue H (to contain the segments' endpoints sorted by x-coordinates)
- Repeat
  - Take the next "event" p from H:

  // Update V
  - If p is the left endpoint of a segment, add the segment to V
  - If p is the right endpoint of a segment, remove the segment from V
  - If p is the intersection point of s and s', swap the order of s and s' in V, report p

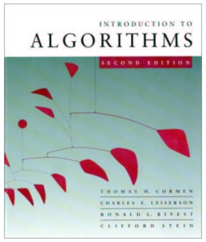
Algorithm ctd.

// Update H

– For each new pair of neighbors s and s’ in V:

- Check if s and s’ intersect on the right side of the sweep line
- If so, add their intersection point to H
- Remove the possible duplicates in H

- Until H is empty



Analysis

- Initializing H: O(n log n)
- Updating V:
  - O(log n) per operation
  - O((P+n) log n) total
- Updating H:
  - O(log n) per intersection
  - O(P log n) total
- Overall: O((P+ n) log n) time

Correctness

- All reported intersections are correct
- Assume there is an intersection not reported. Let p=(x,y) be the first such unreported intersection (of s and s')
- Let x' be the last event before p. Observe that:
  - At time x' segments s and s' are neighbors on the sweep line
  - Since no intersections were missed till then, V maintained the right order of intersecting segments
  - Thus, s and s' were neighbors in V at time x'. Thus, their intersection should have been detected