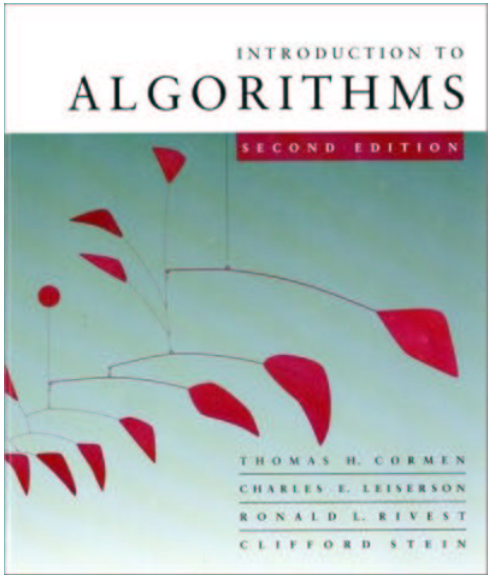
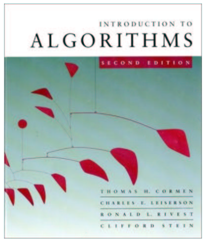
Introduction to Algorithms

6.046J/18.401



Lecture 17

Prof. Piotr Indyk

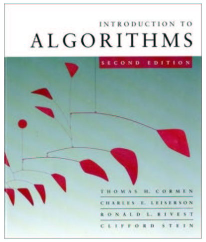


Fast Fourier Transform

- Discrete Fourier Transform (DFT):
  - Given: coefficients of a polynomial

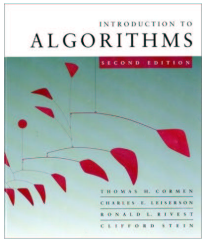
    $$a(x)=a_0+a_1x+\ldots+a_{n-1}x^{n-1}$$

  - Goal: compute $a(\omega_n^0), a(\omega_n^1) \ldots a(\omega_n^{n-1})$, ω_n is the "principal n-th root of unity"
- Challenge: Perform DFT in $O(n \log n)$ time.



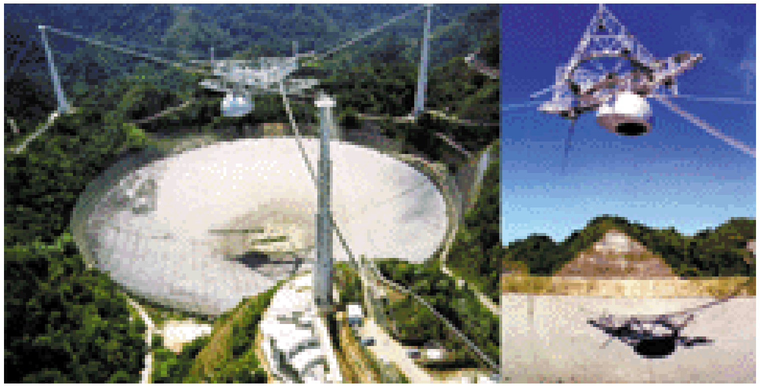
Motivation I: 6.003

- FFT is essential for digital signal processing
  - $a_0, a_1, \dots, a_{n-1}$: signal in the "time domain"
  - $a(\omega_n^0), a(\omega_n^1) \dots a(\omega_n^{n-1})$: signal in the "frequency domain"
  - FFT enables quick conversion from one domain to the other
- Used in Compact Disks, Digital Cameras, Synthesizers, etc, etc.

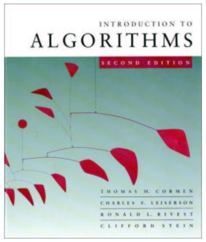


Example application: SETI

- Searching For Extraterrestial Intelligence (SETI):

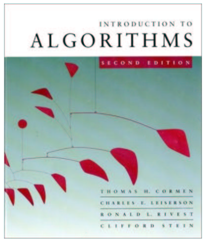


"At each drift rate, the client searches for signals at one or more bandwidths between 0.075 and 1,221 Hz. This is accomplished by using FFTs of length 2^n ($n = 3, 4, ..., 17$) to transform the data into a number of time-ordered power spectra."



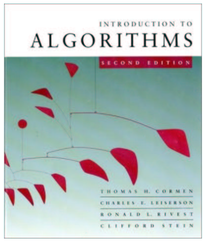
FFT

- Very elaborate implementations (e.g., FFTW, “the Fastest Fourier Transform in the West”, done at MIT)
- Hardware implementations



Motivation II: Computer Science

- We will see how to multiply two polynomials in O(n log n) time using FFT
- Multiplication of polynomials → mult. of (large) integers - cryptography
- Also: pattern matching, etc.


INTRODUCTION TO
ALGORITHMS
SECOND EDITION
THOMAS H. CORMEN
CHARLES E. LEISERSON
RONALD L. RIVEST
CLIFFORD STEIN

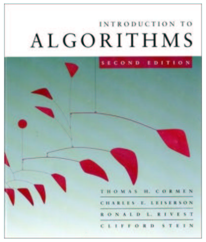
DFT

- Recall: want $a(\omega_n^0)$, $a(\omega_n^1)$ … $a(\omega_n^{n-1})$
- ω_n is the “principal n-th root of unity, i.e., for $j=0\ldots n-1$ we have $(\omega_n^j)^n=1$
- We will work in the field of complex numbers where

$$\omega_n = e^{2\pi i/n} = \cos(2\pi/n) + i\sin(2\pi/n)$$

- ω_n is indeed the principal n-th root of unity:

$$(\omega_n^j)^n = e^{2\pi ij} = \cos(2\pi j) + i\sin(2\pi j) = 1$$

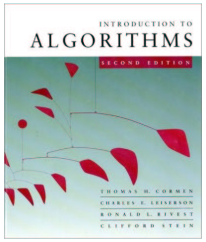


Halving Lemma

- If $n>0$ is even, then the squares of the n complex n-th roots of unity are the $n/2$ complex $(n/2)$-th roots of unity, i.e.:

$$\{ (\omega_n^0)^2, \ldots , (\omega_n^{n-1})^2\} = \{\omega_{n/2}^0, \ldots , \omega_{n/2}^{n/2-1}\}$$

- Proof: $(\omega_n^j)^2 = e^{2(2\pi ij/n)} = e^{2\pi ij/(n/2)} = \omega_{n/2}^j$



FFT

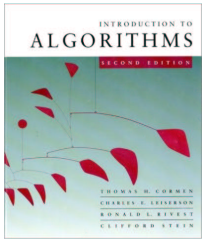
- Divide-and-conquer algorithm
- "Split" $a(x)$ into $a^{[0]}(x)$ and $a^{[1]}(x)$:

$$a^{[0]}(x)= a_0+a_2x+\ldots +a_{n-2}x^{n/2-1}$$

$$a^{[1]}(x)= a_1+a_3x+\ldots +a_{n-1}x^{n/2-1}$$

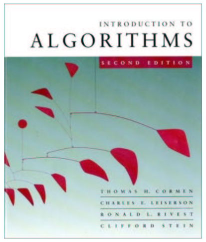
- Therefore

$$a^{[0]}(x^2)+ x\, a^{[1]}(x^2) = a(x)$$



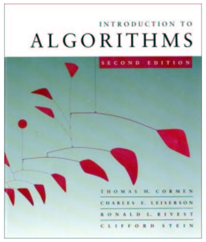
FFT: the algorithm

- Recall we need to evaluate the polynomial a at points $\{\omega_n^0, \ldots, \omega_n^{n-1}\}$
- Suffices to
  - – Evaluate polynomials $a^{[0]}$ and $a^{[1]}$ at points $\{(\omega_n^0)^2 \ldots, (\omega_n^{n-1})^2\}=P$
  - – Compute $a(\omega_n^j) = a^{[0]}((\omega_n^j)^2) + \omega_n^j a^{[1]}((\omega_n^j)^2)$
- However, $P= \{\omega_{n/2}^0, \ldots, \omega_{n/2}^{n/2-1}\}$, $|P|=n/2$
- Thus, we just need to recursively evaluate two polynomials with degree $n/2-1$ at $n/2$ points!
- Time: $T(n)=2\,T(n/2) + O(n) \rightarrow T(n)=O(n \log n)$



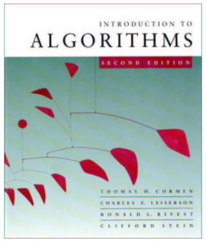
Comments

- We assumed that n is a power of 2
- This is NOT without loss of generality



Inverse DFT

- Given: the values $a(\omega_n^0)$, $a(\omega_n^1)$ … $a(\omega_n^{n-1})$, denoted by $y_0, y_1, \dots, y_{n-1}$.
- Goal: compute the coefficients $a_0, a_1, \dots, a_{n-1}$
- Algorithm:
  - “Observe” that $a_j = y((\omega_n^{-1})^j)$, $y(x)$ is a polynomial with coefficients $y_0, \dots, y_{n-1}$ (see CLRS for proof)
  - Run FFT



Polynomial multiplication

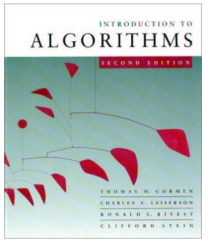
Input: $a(x)=a_0+a_1x+\ldots+a_{n-1}x^{n-1}$,

$b(x)=b_0+b_1x+\ldots+b_{n-1}x^{n-1}$,

Output: $c(x)=a(x)*b(x)=c_0+c_1x+\ldots+c_{2n-2}x^{2n-2}$

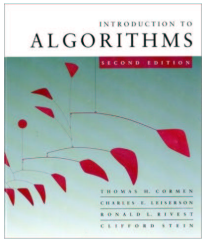
$c_i=a_0b_i+a_1b_{i-1}+\ldots+a_{i-1}b_1+a_ib_0$

How to solve it in $O(n \log n)$ time ?



FFT-based algorithm

- Extend a,b to degree $2n-2$ (by adding 0’s)
- Compute $a(\omega_{2n}^{0})\ldots a(\omega_{2n}^{2n-2})$ and
  $b(\omega_{2n}^{0})\ldots b(\omega_{2n}^{2n-2})$ (via FFT)
- Compute $c(\omega_{2n}^{j})= a(\omega_{2n}^{j})*b(\omega_{2n}^{j}), j=0\ldots 2n-2$
- Compute $c_0, c_1,\ldots,c_{2n-2}$ (via inverse FFT)
- Same time as FFT



Uniqueness of c

- Can show (CLRS) that if we fix the values of a (d-1)-degree polynomial at d different points, then the polynomial is unique
- E.g., there is only one line passing through 2 points
- Therefore, the algorithm is correct