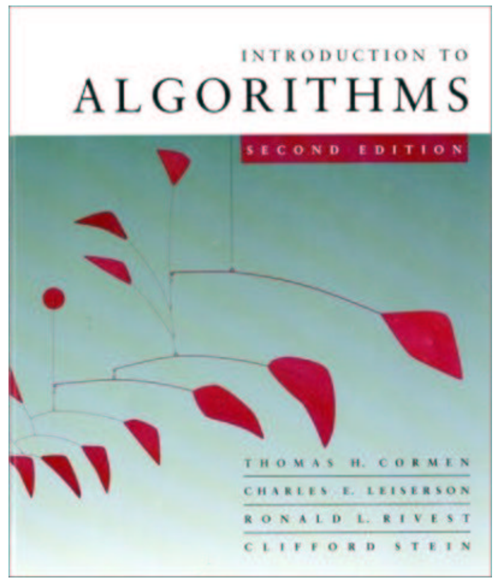
# *Introduction to Algorithms*

## 6.046J/18.401J




# *Lecture 16*

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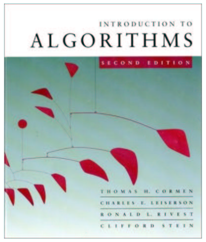
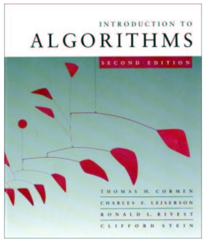


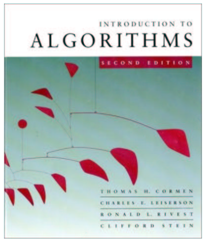

# String Matching

- Input: Two strings T[1…n] and P[1…m], containing symbols from alphabet $\Sigma$
- Goal: find all "shifts" $1 \le s \le n-m$ such that T[s+1…s+m]=P
- Example:
  - $\Sigma$={ ,a,b,…,z}
  - T[1…18]="to be or not to be"
  - P[1..2]="be"
  - Shifts: 3, 16



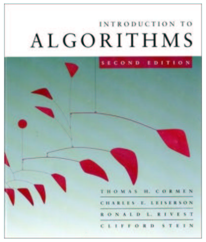

# Simple Algorithm

```
for s ← 0 to n-m
    Match ← 1
    for j ← 1 to m
        if T[s+j] ≠ P[j] then
            Match ← 0
            exit loop
    if Match=1 then output s
```




# Results

- Running time of the simple algorithm:
  - – Worst-case: O(nm)
  - – Average-case (random text): O(n)
- Is it possible to achieve O(n) for any input ?
  - – Knuth-Morris-Pratt'77: deterministic
  - – Karp-Rabin'81: randomized

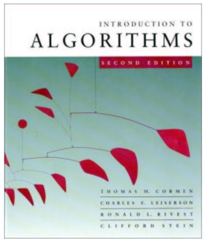



# Karp-Rabin Algorithm

- A very elegant use of an idea that we have encountered before, namely…

  HASHING !
- Idea:
  – Hash all substrings T[1…m], T[2…m+1], T[3…m+2], etc.
  – Hash the pattern P[1…m]
  – Report the substrings that hash to the same value as P

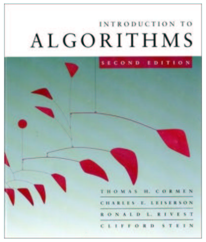
- Problem: how to hash n-m substrings, each of length m, in O(n) time ?




# Implementation

- Attempt I:
  - Assume $\Sigma=\{0,1\}$
  - Think about each $T^s=T[s+1\ldots s+m]$ as a number in binary representation, i.e.,

    $$t_s=T[s+1]2^0+T[s+2]2^1+\ldots+T[s+m]2^{m-1}$$
  - Find a fast way of computing $t_{s+1}$ given $t_s$
  - Output all $s$ such that $t_s$ is equal to the number $p$ represented by $P$




# The great formula

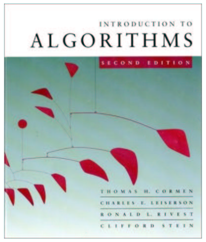
- How to transform

$$t_s = T[s+1]2^0 + T[s+2]2^1 + \ldots + T[s+m]2^{m-1}$$

into

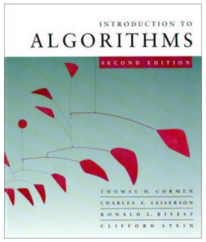
$$t_{s+1} = T[s+2]2^0 + T[s+3]2^1 + \ldots + T[s+m+1]2^{m-1} \text{ ?}$$

- Three steps:
  - Subtract $T[s+1]2^0$
  - Divide by $2$ (i.e., shift the bits by one position)
  - Add $T[s+m+1]2^{m-1}$
- Therefore: $t_{s+1} = (t_s - T[s+1]2^0)/2 + T[s+m+1]2^{m-1}$



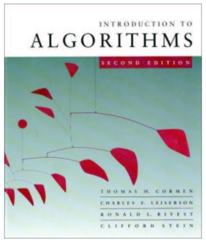

# Algorithm

- Can compute $t_{s+1}$ from $t_s$ using 3 arithmetic operations
- Therefore, we can compute all $t_0, t_1, \ldots, t_{n-m}$ using $O(n)$ arithmetic operations
- We can compute a number corresponding to $P$ using $O(m)$ arithmetic operations
- Are we done ?




# Problem

- To get O(n) time, we would need to perform each arithmetic operation in O(1) time
- However, the arguments are m-bit long !
- It is unreasonable to assume that operations on such big numbers can be done in O(1) time
- We need to reduce the number range to something more managable




# Hashing

- We will instead compute

$$t'_s = T[s+1]2^0 + T[s+2]2^1 + \ldots + T[s+m]2^{m-1} \bmod q$$

where $q$ is an “appropriate” prime number

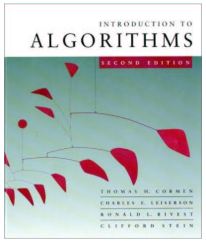
- One can still compute $t'_{s+1}$ from $t'_s$ :

$$t'_{s+1} = (t'_s - T[s+1]2^0) * 2^{-1} + T[s+m+1]2^{m-1} \bmod q$$

- If q is not large, i.e., has $O(\log n)$ bits, we can compute all $t'_s$ (and $p'$) in $O(n)$ time

# Problem

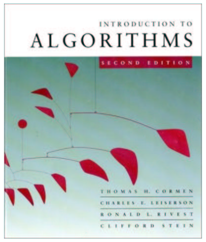
- Unfortunately, we can have false positives, i.e., $T^s \neq P$ but $t'_s = p'$
- Need to use a random $q$
- We will show that the probability of a false positive is small → randomized algorithm




# False positives

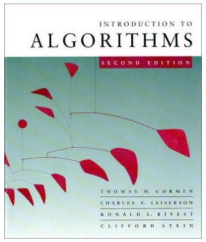
- Consider any $t_s \neq p$. We know that both numbers are in the range $\{0\ldots2^m-1\}$
- How many primes $q$ are there such that

  $t_s \bmod q = p \bmod q \quad \equiv \quad (t_s-p) = 0 \bmod q$ ?
- Such prime has to divide $x=(t_s-p) \leq 2^m$
- Represent $x=p_1^{e1}p_2^{e2}\ldots p_k^{ek}$, $p_i$ prime, $e_i \geq 1$
- Since $2 \leq p_i$ , we have $2^k \leq x \leq 2^m \rightarrow k \leq m$
- There are $\leq m$ primes dividing $x$



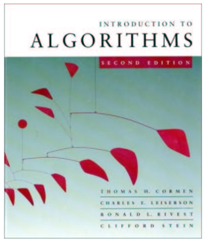

# Algorithm

- Let $\prod$ be a set of $2nm$ primes, each having $O(\log n)$ bits
- Choose $q$ uniformly at random from $\prod$
- Compute $t'_0, t'_1, \ldots.,$ and $p'$
- For each $s$, the probability that $t'_s = p'$ while $T^s \neq P$ is at most $m/2nm = 1/2n$
- The probability of *any* false positive is at most $(n-m)/2n \leq 1/2$




# "Details"

- How do we know that such $\Pi$ exists ?
- How do we choose a random prime from $\Pi$ in $O(n)$ time ?



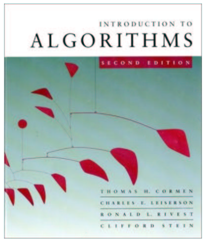

# Prime density

- Primes are "dense". I.e., if PRIMES(N) is the set of primes smaller than N, then asymptotically

$$|\mathrm{PRIMES}(N)|/N \sim 1/\log N$$

- If N large enough, then

$$|\mathrm{PRIMES}(N)| \geq N/(2\log N)$$

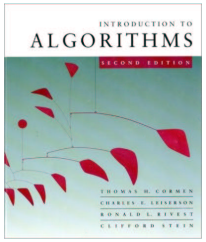



# Prime density continued

- If we set $N=9mn \log n$, and $N$ large enough, then

$$|PRIMES(N)| \geq N/(2\log N) \geq 2mn$$

- All elements of $PRIMES(N)$ are $\log N = O(\log n)$ bits long




# Prime selection

- Still need to find a random element of PRIMES(N)
- Solution:
  – Choose a random element from {1 … N}
  – Check if it is prime
  – If not, repeat

# Prime selection analysis

- A random element q from {1…N} is prime with probability ~1/log N
- We can check if q is prime in time polynomial in log N (trust me J )
- Therefore, we can generate random prime q in o(n) time
- The rest of the algorithm takes O(n) time