Lecture 24: May 9, 2002

Today:

- NP-completeness continued (connecting more things)
In the previous episode

• NP - problems whose solutions are verifiable in polynomial time
• Poly time reductions
• Any problem in NP can be reduced to SAT, SAT in NP ⇒ SAT NP-complete
• SAT → 3SAT, 3SAT in NP ⇒ 3SAT NP-complete
Today’s menu

- SAT → Clique
- Clique → Independent Set
- Independent Set → Vertex Cover
- Vertex Cover → Set Cover
- Set Cover → Dominating Set

Therefore, all of them are NP-hard (and NP-complete).
SAT $\rightarrow$ Clique

Clique:

- **Input**: Undirected graph $(V, E)$, $K > 0$.
- **Output**: Is there $C \subseteq V$ of size $\geq K$ such that every pair of vertices in $C$ has an edge between them.

SAT:

- **Input**: $n$ Boolean variables $x_1, \ldots, x_n$ and a formula $\phi$ with $m$ clauses, e.g.,

$$x_1 \lor x_2 \lor x_5, \ x_1 \lor \neg x_2, \ldots$$

- **Goal**: Check if there exists a TRUE/FALSE assignments of variables such that every clause is satisfied, i.e., every clause above has one term set to TRUE.

Our goal: given a SAT formula $\phi = C_1 \ldots C_m$, produce $G = (V, E)$ and $K$, such that $\phi$ satisfiable if and only if $G$ has clique of size $\geq K$. 
Reduction

Glossary: a literal is either $x_i$ or $\neg x_i$.

Reduction:

- For each literal $l_j$ in $\phi$, create a vertex $v_j$
- Create an edge $v_j - v_{j'}$ if $\neg v_j, v_{j'}$ do not correspond to literals from the same clause
  $\neg l_j$ is not a negation of $l_{j'}$

E.g., the formula $x_1 \lor x_2 \lor x_3$, $\neg x_2 \lor \neg x_3, \neg x_1 \lor x_2$ is transformed into graph

Claim: $\phi$ satisfiable if and only if clique $\geq m$. 
Independent set

Independent set (IS):

- Input: Undirected graph \((V, E), K > 0\).
- Output: Is there \(S \subseteq V\) of size \(\geq K\) such that every pair of vertices in \(S\) has no edge between them.

Clique for \((V, E) \rightarrow IS for (V', E')\):

- Keep the same vertex set, i.e., \(V' = V\)
- Take \(E'\) to be the complement of \(E\)

Claim: any clique \(C\) in \((V, E)\) is an independent set in \((V', E')\).

Corollary: \((V, E)\) has clique \(\geq K\) if and only if \((V', E')\) has independent set \(\geq K\).
**Vertex cover**

Vertex cover (VC):

- **Input**: Undirected graph \((V, E)\), \(L > 0\).
- **Output**: Is there \(C \subseteq V\) of size \(\leq L\) such that every edge in \(E\) has at least one endpoint in \(C\) ?

**IS for \((V, E) \rightarrow VC for \((V', E')\):**

- Keep the same vertex set \((V' = V)\)
- Keep the same edge set \((E' = E)\)
- Set \(L = n - K\)

Claim: \(S\) is an independent set in \((V, E)\) if and only if the *complement* of \(S\) is a vertex cover for \((V', E')\).

Corollary: \((V, E)\) has independent set \(\geq K\) if and only if \((V', E')\) has vertex cover \(\leq n - K\).
Set cover

Set cover (SC):

- **Input**: a family $\mathcal{S}$ of sets $S_1 \ldots S_n \subset U$, $L > 0$.
- **Output**: Is there $I \subset \{1 \ldots n\}$, of size $\leq L$ such that

$$\bigcup_{i \in I} S_i = U$$

VC for $(V, E) \rightarrow$ SC for $\mathcal{S}, U$:

- Set $U = E$
- For each vertex $v \in V$, set $S_v$ to be the set of edges incident to $v$

Claim: $C$ is a vertex cover for $(V, E)$ if and only if $I = C$ covers $E$. 

Dominating set

Dominating set (DS):

- Input: a graph $G = (V, E)$, $L > 0$.
- Output: Is there $S \subseteq V$, of size $\leq L$, which dominates $V$, i.e., such that any vertex is either in $S$ is adjacent to a vertex in $S$?

SC for $S, U \rightarrow$ DS for $(V, E)$. 
SC to DS

Proof by example:

- Universe $U = \{u_1, u_2, u_3, u_4\}$
- Sets $\{u_1, u_3\}, \{u_1, u_4\}, \{u_2, u_3, u_4\}$

We create a graph $(V, E)$ as follows:

Claim: There is a set cover of size $\leq L$ if and only if there is a dominating set of size $\leq L + 1$. 
Proof

• Set cover \( I \to \) dominating set \( S \):
  – Take all vertices corresponding to sets in \( I \)
  – Take vertex \( u \)
  – We get a dominating set \( S \) of size \(|I| + 1\)
• Dominating set \( S \to \) set cover \( I \)
  – Transform the dominating set so that
    * Vertex \( u \) is in \( S \). To this end, observe that either \( u \) or \( v \) must be in \( S \). If \( v \) is in \( S \) and \( u \) is not, we remove \( v \) and add \( u \). The result is still a dominating set.
    * \( S \) does not contain any \( u_i \). To this end, for any \( u_i \in S \) we
      - Remove \( u_i \) from \( S \)
      - Add any vertex \( S_j \) with an edge to \( u_i \)
    The resulting set is still a dominating set.
  – The set cover \( I \) is defined by nodes \( S_i \in S \).