Lecture 20: May 3, 2001

Computational Geometry II

- Range queries continued

Orthogonal range queries

- Given: a set of points $p_1 \dots p_n$
- Goal: preprocess the points so that later, given a square $T = [x_l, x_r] \times [y_l, y_r]$, we can quickly check if there is $p_i \in T$

We ignore the preprocessing time, for simplicity.

Discretization

In general, coordinates of p_i's and T's are reals.

But during last lecture, we showed it suffices to assume they are integers from $\{1 \dots n\}$, by adding

- $O(n)$ space (to store a successor data structure)
- $O(\log n)$ (to "discretize" the rectangle T by finding successors)

From now on, can assume coordinates are from $\{1 \dots n\}$

Solution I

Idea: Since all four parameters of a query T are numbers from $\{1 \dots n\}$, there are at most n^4 queries. Therefore, we can precompute answers to *all* queries and store it in a lookup table. This gives:

- $O(1)$ query time (plus the $O(\log n)$ discretization time)
- $O(n^4)$ space (to store all answers)

Query time is good, but space is *huge*, especially when compared to, say, Binary Search Trees (n space, $\log n$ time). Will try to get the space down.

Solution II

Ideas:

- Decompose the data set into smaller sets, each equippe with a data structure solving "simpler" queries
- Each query can be answered using $O(\log n)$ "simpler" queries
- The total space used by all data structures is small

Now we just need to implement it...

Data structure for simpler queries

A query T is *simple* if it is of the form

$$\{-\infty \ldots \infty\} \times \{y_l \ldots y_r\}$$

I.e., does not specify any constraints on x-coordinates.

But then, the problem is just 1-dimensional, since only the y-coordinates matter !

How to solve it:

- Take the y-coordinates $y_1 \ldots y_m$
- Build a successor-supporting data structure
- To check if there is any y_i falling into the query interval $\{y_l \ldots y_r\}$, check if the successor of y_l is $\leq y_r$

Data structure for simpler queries, ctd.

The structure requires

- $O(\log m)$ query time (when Binary Search Trees are used)
- $O(m)$ space

Now we just need to "lift" the data structure to higher dimensions....

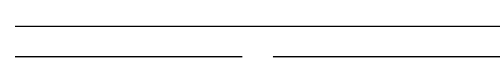
Dyadic intervals

Consider an interval $\{1 \ldots n\}$, assume $n = 2^k$. A *dyadic* interval is:

- the interval $\{1 \ldots n\}$
- intervals I_l and I_r obtained by splitting a dyadic interval into two equal parts. I.e., if $I = \{a \ldots b\}$, then
$$I_l = \{a \ldots (a+b-1)/2\}$$
and
$$I_r = \{(a+b+1)/2 \ldots b\}$$

The length of a dyadic interval is always a power of 2.

Example dyadic intervals:



et cetera

The power of dyadic intervals

Basic fact of life: any interval $I = \{a \dots b\} \subset \{1 \dots n\}$ can be decomposed into a union of $\leq 2 \log n$ dyadic intervals.

"Proof": For simplicity, focus on intervals of the form $\{1 \dots y\}$, $y < n$. Run the following algorithm (similar to binary search):

1. Set $L = 1, R = n$
2. While $L \neq R$
   - Compute $M = (l + r - 1)/2$
   - If $M \leq y$, report interval $\{L \dots M\}$ and set $L = M + 1$
   - Otherwise, set $R = M$

For general intervals algorithm is similar but a bit more complex.

The decomposition of point-set

- For any $i = 1 \dots \log n$
  - For any dyadic interval I of length 2^i, construct the set of points $P_I = P \cap (I \times \{1 \dots n\})$
  - Prepare a data structure solving simple queries for the set P_I

The decomposition of query rectangle

In order to answer query $T = I_x \times I_y$:

- Decompose the interval I_x into $\leq 2 \log n$ dyadic intervals $I_1 \dots I_l$
- For each such interval I_i, query the data structure designed for I_i with a query $\{-\infty, \infty\} \times I_y$
- If any query returned YES, return YES. Otherwise, return NO

Complexity

- $2 \log n \cdot O(\log n)$ query time, since each simple query takes $O(\log n)$ time and we perform $\leq 2 \log n$ of them
- $O(n \log n)$ space

Structure	Space	Time
Do nothing	$O(n)$	$O(n)$
Solution I	$O(n^4)$	$O(\log n)$
Solution II	$O(n \log n)$	$O(\log^2 n)$

Solution II is "the best of both worlds".

The query time can be made $O(\log n)$, but it is difficult.