Lecture 3: February 13, 2001

Today:

- Divide-and-conquer paradigm
- Strassen's algorithm
- Polynomial multiplication

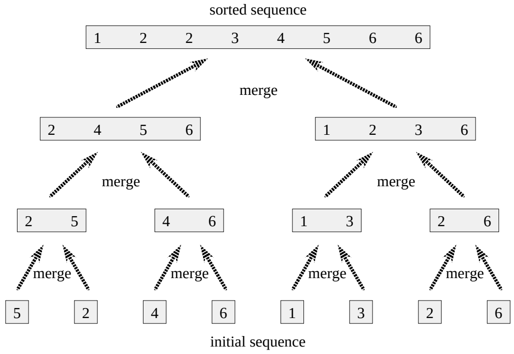
Merge Sort (review)

To sort n numbers:

1. if n=1, done.
2. recursively sort 2 lists of numbers with $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ elements
3. merge 2 sorted lists in $\Theta(n)$ time

Structure of merge sort algorithm

- break problem into similar (smaller) subproblems
- recursively solve subproblems
- combine solutions to produce final answer



Divide-and-conquer paradigm

1. <u>Divide</u> problem into subproblems.
2. <u>Conquer</u> subproblems by solving recursively.
3. <u>Combine</u> subproblem solutions.

Merge sort as Divide-and-conquer algorithm

1. **Divide:** Divide n-array into two $n/2$-subarrays.
2. **Conquer:** Sort the two subarrays recursively.
3. **Combine:** Linear-time merge.

Recurrence for Merge sort

$$T(n) = \underbrace{2}_{\#\text{ subproblems}} T(\overbrace{n/2}^{\text{subproblem size}}) + \underbrace{\Theta(n)}_{\text{work dividing \& combining}}$$

$$T(n) = 2T(n/2) + \Theta(n)$$

Solve recurrence

review from last lecture
Master Theorem: $T(n) = aT(n/b) + f(n)$

1. $f(n) = O(n^{\log_b a - \epsilon}) \Rightarrow$
   $T(n) = O\left(n^{\log_b a}\right)$
2. $f(n) = \Theta(n^{\log_b a}) \lg^k n \ k \geq 0 \Rightarrow$
   $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$
3. $f(n) = \Omega(n^{\log_b a + \epsilon})$ and $af(n/b) \leq cf(n)$
   $\Rightarrow \quad T(n) = \Theta(f(n))$

Merge sort analysis

- write recurrence $T(n) = 2 \cdot T(n/2) + \Theta(n)$
- solve using MT

  $a = 2$ and $b = 2 \ \Rightarrow \quad n^{\log_2 2} = n \ \Rightarrow$

  Case 2 of MT ($k = 0$) $\Rightarrow$

  $T(n) = \Theta(n \lg n)$

Binary search

Search a key in a sorted array

- Example Given array:

3 5 7 8 9 12 15

Find key **9**

Binary search — Divide-and-conquer algorithm

to find key in sorted array of n elements:

1. **Divide:** Check middle element.
2. **Conquer:** Search one subarray.
3. **Combine:** Trivial.

Recurrence for binary search

halve n on each iteration $\Rightarrow$
\# times halve n to get 1 is $\lg n$

$$T(n) = \underbrace{1}_{\#\text{ subproblems}} T(\overbrace{n/2}^{\text{subproblem size}}) + \underbrace{\Theta(1)}_{\text{work dividing \& combining}}$$

$$T(n) = T(n/2) + \Theta(1)$$

Solve recurrence using MT

$a = 1$ and $b = 2 \Rightarrow n^{\log_2 1} = 0 \Rightarrow$

Case 2 of MT ($k = 0$) $\Rightarrow$

$T(n) = \Theta(\lg n)$

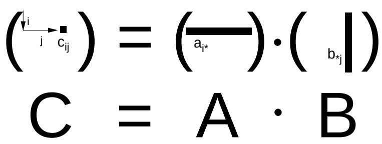
Matrix multiplication

$A,\ B,\ C$ — $n \times n$ matrices

Input: $A = (a_{ij}),\quad B = (b_{ij})$

Output: $C = (c_{ij}) = AB$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \text{ for } i, j = 1, \ldots, n$$



Time to compute $c_{ij} \ = \ \Theta(n)$

Time to compute $C \ = \ \Theta(n) \cdot n^2 = \Theta(n^3)$

Divide-and-conquer algorithm

$n \times n$ matrix$=2 \times 2$ matrix of $n/2 \times n/2$ submatrices

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & g \\ f & h \end{pmatrix}$$

$$C \qquad = A \qquad \times\ B$$

A, B, C — $n \times n$ matrices
$a, b, ..., f, h, ..., t, u$ all $n/2 \times n/2$ matrices

Reduction to subproblems of size $n/2 \times n/2$

We can rewrite the equation

$$C = A \times B$$

as

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & g \\ f & h \end{pmatrix}$$

so that

$$\begin{aligned} r &= ae + bf \\ s &= ag + bh \\ t &= ce + df \\ u &= cg + dh \end{aligned}$$

One $n \times n$ matrix multiplication is reduced to four equations, each with:

- 2 multiplications of $n/2 \times n/2$ matrices
- 1 addition of $n/2 \times n/2$ matrices

$\Rightarrow$ Total: 8 Mults, 4 Adds

Analysis

(n is the dimension of matrix)

$$T(n) = \underbrace{8}_{\#\text{ subproblems}} T(\overbrace{n/2}^{\text{subproblem size}}) + \underbrace{\Theta(n^2)}_{\text{work dividing \& combining}}$$

$$T(n) = 8T(n/2) + \Theta(n^2)$$

$n^{\log_2 8} = n^3 \Rightarrow$ Case 1 of MT

$T(n) = \Theta(n^3)$ No better!

Recursive algorithm

Idea: Multiply 2×2 matrices with only 7 scalar multiplications (instead of 8)
Let

$$\begin{aligned}
P_1 &= a \cdot (g - h) \\
P_2 &= (a + b) \cdot h \\
P_3 &= (c + d) \cdot e \\
P_4 &= d \cdot (f - e) \\
P_5 &= (a + d) \cdot (e + h) \\
P_6 &= (b - d) \cdot (f + h) \\
P_7 &= (a - c) \cdot (e + g)
\end{aligned}$$

then

$$\begin{aligned}
r &= P_5 + P_4 - P_2 + P_6 \\
s &= P_1 + P_2 \\
t &= P_3 + P_4 \\
u &= P_5 + P_1 - P_3 - P_7
\end{aligned}$$

Indeed,

$$\begin{aligned}
s &= P_1 + P_2 \\
&= (ag - ah) + (ah + bh) \\
&= ag + bh
\end{aligned}$$

Check the rest!

Do not rely on commutativity of multiplication!
Same equations true for matrices instead of scalars.

Old method: 8 Mults, 4 Adds
New method: To compute $P_1, ..., P_7, r, s, t, u$ need

- 7 multiplications
- 18 additions (and subtractions)

For $n \times n$ matrices multiplication takes $\Theta(n^3)$, but addition takes $\Theta(n^2)$.

14 more adds! — cost of eliminating 1 multiply
For scalars, not worth it.
For matrices, can be worth it.

Strassen's algorithm

to multiply $n \times n$ matrices $C = AB$

1. **Divide:** Partition A, B into $n/2 \times n/2$ matrices. Use $+$ and $-$ to form terms to be multiplied.
2. **Conquer:** Perform 7 multiplications recursively.
3. **Combine:** Form C using $+$ and $-$.

Analysis

$$T(n) = \underbrace{7}_{\#\text{ multiplications}} T(\overbrace{n/2}^{\text{subproblem size}}) + \underbrace{\Theta(n^2)}_{\text{work combining, i.e. adding}}$$

$$\begin{aligned} T(n) &= 7T(n/2) + \Theta(n^2) \\ &= \Theta(n^{\lg 7}) \quad \text{(Case 1 of MT)} \\ &= O(n^{2.81}) \end{aligned}$$

$\Theta(n^{2.81})$ vs $\Theta(n^3)$, wins for $n \geq 50$ in practice

Best algorithm to date: $O(n^{2.376})$ (not practical!)

Polynomial Multiplication

A, B —n-th order poly, C — $2n$-th order poly

$$\begin{aligned} A(x) &= a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \\ B(x) &= b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n \\ C(x) &= A(x) \cdot B(x) \\ &= c_0 + c_1 x + c_2 x^2 + \ldots + c_{2n} x^{2n} \\ c_i &= \sum_{k=0}^{i} a_k b_{i-k} \quad \text{for } i = 0, .., 2n \end{aligned}$$

Convention: $a_j, b_j \equiv 0$ when $j > n$

Ordinary algorithm: $\Theta(n^2)$ running time.

Polynomial Multiplication

$$c_i = \sum_{k=0}^{i} a_k b_{i-k} \quad \text{for } i = 0, .., 2n$$

Example: $n = 2$

$$\begin{aligned} A(x) &= a_0 + a_1 x + a_2 x^2 \\ B(x) &= b_0 + b_1 x + b_2 x^2 \\ C(x) &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 \end{aligned}$$

$$\begin{aligned} c_0 &= \sum_{k=0}^{0} a_k b_{0-k} = a_0 b_0 \\ c_1 &= \sum_{k=0}^{1} a_k b_{1-k} = a_0 b_1 + a_1 b_0 \\ &= \cdots \\ c_4 &= \sum_{k=0}^{4} a_k b_{4-k} = a_2 b_2 \end{aligned}$$

Write C in $n/2$-th order polynomials

Factor $x^{n/2}$ out of latter terms of A, B:

$$\begin{aligned} A(x) &= (a_0 + ... + a_{n/2-1}x^{n/2-1}) + \\ &\quad x^{n/2}(a_{n/2} + a_{n/2+1}x + ... + a_n x^{n/2}) \\ &= P(x) + x^{n/2}Q(x) \\ B(x) &= R(x) + x^{n/2}S(x) \end{aligned}$$

$$\begin{aligned} C(x) &= P(x) \cdot R(x) \\ &\quad + x^{n/2}(P(x) \cdot S(x) + R(x) \cdot Q(x)) \\ &\quad + x^n Q(x) \cdot S(x) \end{aligned}$$

Divide-and-conquer algorithm

1. **Divide:** Partition A, B into sum of a $n/2$-th order poly and $x^{n/2}\times$ ($n/2$-th order poly).
2. **Conquer:** Perform 4 multiplications ($P \cdot R$, $P \cdot S, R \cdot Q$, and $Q \cdot S$) recursively.
3. **Combine:** Form C using $+$ and $-$.

Analysis:

$$T(n) = 4T(n/2) + \Theta(n)$$

$$n^{\log_2 4} = n^2 \Rightarrow \text{Case 1 of MT}$$

$$T(n) = \Theta(n^2) \qquad \text{No better!}$$

Clever divide-and-conquer

$$(a + by)(c + dy) = \overbrace{ac + (ad + bc)y + bdy^2}^{\text{4 Mults!}}$$

Let

$$\begin{aligned} m_1 &= (a + b) \cdot (c + d) \\ m_2 &= a \cdot c \\ m_3 &= b \cdot d \end{aligned}$$

then don't need 4th multiplication, since:

$$ad + bc = m_1 - m_2 - m_3$$

Let a, b, c, d be polynomials, $y = x^{n/2}$ use divide-and-conquer to perform only **3** multiplications recursively.

Analysis

$$T(n) = 3T(n/2) + \Theta(n)$$

$$n^{\log_2 3} = n^{1.59} \Rightarrow \text{Case 1 of MT}$$

$$T(n) = \Theta(n^{\lg 3})$$