Lecture 1: February 6, 2001

Welcome to 6.046J/18.410:
Introduction to Algorithms

Today's Handouts:

- #1: Course Objectives
- #2: Signup Sheet
- #3: Course Information
- #4: Course Calendar
- #5: Reference List
- #6: Diagnostic Survey
- #7: Kingston (Reading for recitation)

Today:

- Course administration
- Course overview
- A sample algorithm: Insertion Sort.

What is an algorithm?

A sequence of elementary computational steps that transforms the input (of some mathematical function) into the output.

Multiplication

Input: Positive Integers a, b.
Output: $a \times b$.

- Well defined mathematical function.
- How to compute it?

"Algorithms"

Example: $a = 365$, $b = 24$.
Method 1:

$$
\begin{aligned}
365 * 24 &= 365 + (365 * 23) \\
&= 730 + (365 * 22) \\
&\vdots \\
&= 8760 + (365 * 0) \\
&= 8760
\end{aligned}
$$

Method 2:

```
     3 6 5
       2 4
 - - - - -
   1 4 6 0
   7 3 0
 - - - - -
   8 7 6 0
```


"Algorithms"

- Both methods can be abstracted and made to work for any pair of positive integers, yielding "algorithms" for multiplication.
- Thus same "problem" has multiple algorithms.
- Which one is better? How to compare?

  Analysis component of 6.046.
- Are there better algorithms? How to find them?

  Design component of 6.046.
- What else?

  Fundamental problems; Solutions; Limitations

Opening example

Problem: **Sorting**

Input: Sequence of n numbers $\langle a_1, a_2, \ldots, a_n \rangle$.

Output: Permutation $\langle a'_1, a'_2, \ldots, a'_n \rangle$ such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$.

How do we describe an algorithm?

Pseudocode notations

- liberal use of English
- use of indentation for block structure
- employ any clear and concise expressive methods
- typically are not concerned with software engineering issues:
  - error handling
  - data abstraction
  - modularity.

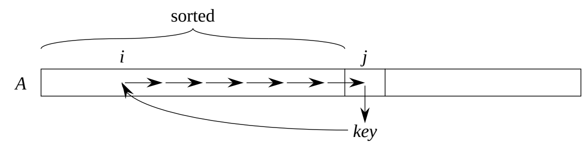
Insertion Sort

Sorts $A[1..n]$ in place

INSERTION-SORT(A)

1 **for** $j \leftarrow 2$ **to** *length*$[A]$
2 **do** $key \leftarrow A[j]$
3 $\triangleright$ Insert $A[j]$ into the sorted sequence $A[1 \,..\, j-1]$.
4 $i \leftarrow j - 1$
5 **while** $i > 0$ and $A[i] > key$
6 **do** $A[i+1] \leftarrow A[i]$
7 $i \leftarrow i - 1$
8 $A[i+1] \leftarrow key$

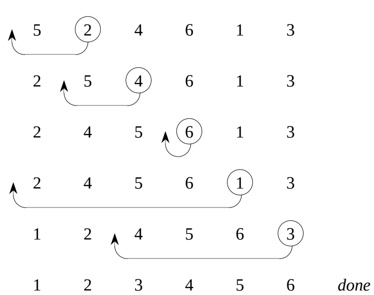
Insertion Sort Idea



$A[1 \;..\; j-1]$ — currently sorted part
$A[j+1 \;..\; n]$ — currently unsorted part

- pick element $A[j]$ (line 2)
- move elements $A[j-1]..A[1]$ to the right until proper position for $A[j]$ is found.

Operation of Insertion Sort Algorithm



Running time

- For a given input, we can compute a running time.
- What do we mean by "time"?
  - CPU? Wall-clock?
  - doesn't it depend on implementation?

Running time

- Depends on
  - input size (e.g. 6 elements vs. 6000)
  - input itself (e.g. partially sorted already)
- Generally want upper bound
- Experiments can suggest performance, but analysis gives...
- Need promise of performance to user

Running time

One natural simplification:

runtime is a fn of size, rather than particular input.

But now there is more than one answer (one for each input).

What to do?

Kinds of analysis

- (usually) – Worst case:
  $T(n)$=max time on *any* input of size n
- (sometimes) – Average case:
  $T(n)$=average time over all inputs of size n (assumes statistical distribution of inputs)
- (never) – Best case: Bad
  Cheat with slow algorithm that works fast on some input. Good only for showing bad lower bound.

For insertion sort, what is worst-case time?

Depends on speed of primitive operations

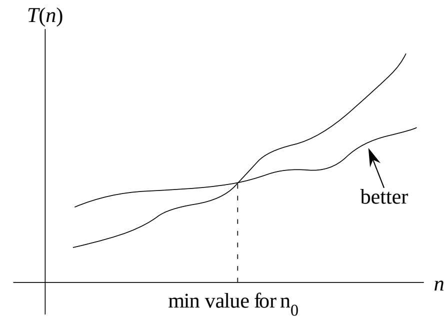
- relative speed (on same machine)
- absolute speed (on different machines)

Asymptotic analysis

- Ignore machine-dependent constants
- Look at *growth* of $T(n)$ as $n \to \infty$
- Θ notation
  - Drop low-order terms
  - Ignore leading constants

$3n^3 + 90n^2 - 2n + 5 = \Theta(n^3)$

As n gets large, a $\Theta(n^2)$ algorithm beats a $\Theta(n^3)$ algorithm.



Insertion Sort Pseudocode

Sorts $A[1..n]$ in place

```
INSERTION-SORT(A)
for j ← 2 to length[A]
    do key ← A[j]
       ▷ Insert A[j] into the sorted
         sequence A[1 . . j − 1].
       i ← j − 1
       while i > 0 and A[i] > key
           do A[i + 1] ← A[i]
              i ← i − 1
       A[i + 1] ← key
```


Insertion Sort Analysis

Write time equations by looking at pseudocode:

Worst case: (input reverse sorted)

Inner loop is $\Theta(j)$ for j from 2 to n:

$$T(n) = \sum_{j=2}^{n} \Theta(j) = \Theta\left(\sum_{j=2}^{n} j\right) = \Theta(n^2)$$

(arithmetic series)

(*Note* casual manipulation of Θ with Σ below.) Why is it OK?

Average case: (all permutations equally likely)
Inner loop is $\Theta(j/2)$

$$T(n) = \sum_{j=2}^{n} \Theta(j/2) = \sum_{j=2}^{n} \Theta(j) = \Theta(n^2)$$

Often, average case not much better than worst case.

Best case: (input already sorted)

$$T(n) = \Theta(n).$$

Is this a fast sorting algorithm?

- Yes, for small n.
- No, for large n.

Why: small leading constant for a $\Theta(n^2)$ algorithm.

Merge Sort

To sort n numbers:

1. if n=1, done.
2. recursively sort 2 lists of numbers with $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ elements, respectively.
3. "merge" 2 sorted lists.

Basic step:

Merge two already sorted lists into one sorted list:

$$\left.\begin{matrix} 2 & 3 & 7 & 8 \\ 1 & 4 & 5 & 6 \end{matrix}\right\} 1\ 2\ 3\ 4\ 5 \ldots$$

Running time of MERGE procedure:

- each step in MERGE takes constant $\Theta(1)$ time
- n such steps

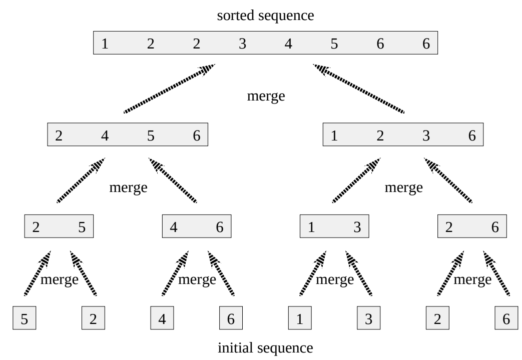
thus, can merge two sorted lists of total length n in $\Theta(n)$ time

Recursive algorithm.

Call MERGE-SORT$(A, 1, n)$ to sort $A[1\,..\,n]$.

MERGE-SORT(A, p, r) $\qquad \triangleright\ T(n)$
  if $p = r$ $\qquad \triangleright\ \Theta(1)$
    then return
  MERGE-SORT$(A, p, \lfloor (p+r)/2 \rfloor)$ $\qquad \triangleright\ 2T(n/2)$
  MERGE-SORT$(A, \lfloor (p+r)/2 \rfloor + 1, r)$
  MERGE results and **return** $\qquad \triangleright\ \Theta(n)$

Operation of Merge Sort



Recurrence

- describes a function recursively in terms of itself
- describes performance of recursive algorithms

Recurrence for merge sort:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1\ , \\ 2T(n/2) + \Theta(n) & \text{if } n > 1\ . \end{cases}$$

$2T(n/2)$ should be $T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil)$,
but this bit of sloppiness turns out not to matter.

(Convention called “integer breakage”.)

How do we find a good upper bound on $T(n)$ in closed form?

- Generally, will assume $\exists c$ s.t. $\forall n < n_0$, $T(n) < c$, i.e. $T(n) = \Theta(1)$ for sufficiently small n.
- Write the recurrence

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1\ , \\ 2T(n/2) + \Theta(n) & \text{if } n > 1\ . \end{cases}$$

  as simply

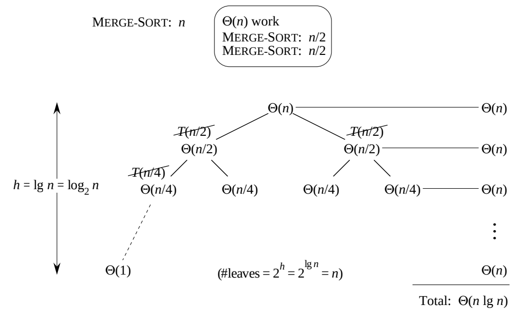
$$T(n) = 2T(n/2) + \Theta(n).$$

Recursion tree

- Model execution to compute running time
- Draw recursion tree, putting in $T()$ at each level then expanding it to $\Theta()$ with $T()$'s under it.

Recursion tree for Merge-Sort

Note: Total implicitly use $(\lg n) \cdot \Theta(n) = \Theta(n \lg n)$



Conclusions

- $\Theta(n \lg n)$ grows more slowly than $\Theta(n^2)$.
  e.g. for $n = 10^6$: $n^2 = 10^{12}$, $n \lg n \approx 2 \cdot 10^7$.
- Therefore merge sort asymptotically beats insertion sort in the worst case.
- In practice, merge sort beats insertion sort for $n > 30$ or so.