Problem Set 7 Solutions

(Exercises were not to be turned in, but we're providing the solutions for your own interest.)

Exercise 7-1. When n is a power of 3, we divide each polynomial into three parts, grouping coefficients for those terms having degrees 0, 1, and 2 mod 3. Formally, $A(x) = A_0(x^3) + xA_1(x^3) + x^2A_2(x^3)$, where A_i has the coefficients of A for only those terms have degrees that are i mod 3. The recurrence for the new algorithm is $T(n) = 3T(n/3) + \Theta(n)$, which by the Master Theorem solves to $T(n) = \Theta(n \log n)$.

Exercise 7-2. The total running time for the ith operations, where i is a power of 2, is $1 + 2 + \cdots + 2^{\lfloor \lg n \rfloor} = 2^{\lfloor \lg n \rfloor + 1} - 1 = \Theta(n)$. The total running time of the other operations is $n - \lfloor \lg n \rfloor$. Therefore the amortized cost per operation is $\Theta(1)$.

Exercise 7-3. The potential function is (a constant multiple c of) the sum of the depths of all the nodes in the heap. We sketch why this works: for INSERT, the actual amount of work done is $\Theta(\log n)$, and the potential function increases by $\Theta(\log n)$ because a new element is added to the tree. For DELETE-MIN, the actual work done is again $\Theta(\log n)$ plus $O(1)$. However, the potential decreases by $c \log n$ because an element is removed. If we choose c to match the constant hidden in the $\Theta(\log n)$, then the decrease in potential cancels out the real work that is done, leaving $\Theta(1)$ amortized cost.

Note that this result is just the result of "clever accounting," and not anything earth-shattering. In any application of a min-heap, the number of INSERT operations must be at least the number of DELETE-MIN operations, so the running time is dominated by the insertions.

Exercise 7-4. To compute the transpose for an adjacency-list representation, we make a new array of adjacency lists for G^T.. We walk down each adjacency list of G. On the list for node u, when encountering a node v, we add u to the front of v's list in G^T. Each step takes $O(1)$ time, so the total time is $O(V + E)$.

For an adjacency-matrix representation, we merely need to compute the transpose matrix. This can be done in $O(V^2)$ time.

Exercise 7-5. (Trivia: this problem is otherwise known as "testing whether a given graph is bipartite.") The wrestlers correspond to nodes in a graph, and their rivalries correspond to edges. Pick an arbitrary vertex s and run a breadth-first search from s to produce a vector d of shortest path lengths from s. (If the graph is unconnected, run BFS on each of its components.) Then iterate over the edges: if (u, v) is an edge and $d[u]$ and $d[v]$ have the same parity (i.e., both even or both odd), then output "no designation." If every edge passes this test, output all u such that $d[u]$ is even as the good guys, and all v such that $d[v]$ is odd as the bad guys.

First, note that if all the edge tests are passed, then the designation is a proper one, because every rivalry is between a good and bad guy. Now suppose some test is not passed for an edge (u, v): in any designation, u and v must be of the same type because they are the same number of "hops" from s. But this means the rivalry between u and v is not satisfied. Thus, there is no valid designation.

The running time is clear: BFS takes linear time $O(n + r)$, and iterating over the edges takes $O(r)$ time, for $O(n + r)$ total.

Exercise 7-6. The graph is on four vertices s, t, u, v, where $w(s, u) = 4$, $w(s, t) = 2$, $w(u, t) = -2$, and $w(t, v) = 1$. Starting from s, we set $d[t] = 2$ and $d[u] = 4$. Therefore t is extracted, so we set $d[v] = d[t] + 1 = 3$. Next v is extracted, and no changes are made to d. Finally u is extracted, and we set $d[t] = d[u] + -2 = 2$, then the algorithm terminates. Note that the shortest path to v is s, u, t, v, and has length 3. However, at the end of the algorithm, $d[v] = 4$ (corresponding to the path s, t, v).

The proof of Theorem 24.6 fails where (on page 598, end of second paragraph) it claims that $\delta(s, y) \leq \delta(s, u)$ "because y occurs before u on a shortest path from s to u *and all edge weights are nonnegative*." In fact, we see in the above example that this is not the case: the shortest path from s to t is s, u, t and has length 2, but the shortest path from s to u has length 4. Therefore the proof of correctness is no longer sound.

Problem 7-1. Maximum Spanning Tree

We note that this problem is very similar to the minimum spanning tree problem. One correct solution involves a direct transformation, by negating all the edge weights of G and running Prim's (or Kruskal's) algorithm on the resulting graph G'. (These algorithms work properly even with negative edge weights.) A minimum spanning tree on G' is a maximum spanning tree on G, because a tree in G' is a tree in G and vice versa, and because the weight of a tree in G' is negated in G.

Another way to solve this problem is by noticing a greedy-choice property, similar to that of the minimum spanning tree (and proven in a very similar way): in any maximum spanning tree T, if we remove an edge (u, v) to yield two trees R, S, then R and S are maximum spanning trees on their respective vertices, and (u, v) is a heaviest edge crossing between those sets of vertices. With this in mind, we can use Prim's algorithm with a *max*-heap, or Kruskal's algorithm with the edges sorted in *descending* order of weights, to find a maximum spanning tree. The running times remain unchanged.

Problem 7-2. Toeplitz Matrices

(a) The sum is Toeplitz. If we are adding matrices A and B (with entries $a_{i,j}$ and $b_{i,j}$, respectively), then the sum C (with entries $c_{i,j}$) has

$$c_{i,j} = a_{i,j} + b_{i,j} = a_{i-1,j-1} + b_{i-1,j-1} = c_{i-1,j-1}$$

as desired.

The product is not necessarily Toeplitz. Here is a counterexample:

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

(b) Note that there are only $2n - 1$ diagonals in an $n \times n$ matrix, and the values on a diagonal are all the same. Therefore we need only a $(2n - 1)$-coordinate vector to represent an $n \times n$ Toeplitz matrix. Specifically, the vector is a tuple of the elements $a_{1,n}, a_{1,n-1}, \ldots, a_{1,1}, a_{2,1}, \ldots, a_{n,1}$. Adding two matrices is done by adding their representative vectors, entry-by-entry. This takes only $O(n)$ time (and space).

(c) Let the input vector be a column vector $\vec{b} = (b_1, \ldots, b_n)^T$, and call the product $\vec{c} = (c_1, \ldots c_n)^T$. Suppose also that we are representing the Toeplitz matrix A by the vector $\vec{a}$ described above. Then by the definition of Toeplitz and matrix multiplication, we have

$$c_i = \sum_{j=1}^{n} a_{n+i-j} b_j = \sum_{j=1}^{2n-1} a_{n+i-j} b_j,$$

where we adopt the convention that $b_j = 0$ when $j > n$, and $a_j = 0$ when $j \leq 0$. But now we see that the coefficient c_i is just the coefficient of the degree-$(n + i)$ term of the product of polynomials a and b, whose representations are given in coefficient form by the vectors $\vec{a}, \vec{b}$. These polynomials have degree $O(n)$, so we can multiply them in $O(n \log n)$ time, as desired.

Problem 7-3. Amortized Queues

(a) The total work is $3 + (6 + 2) + 3 + (1 + 6 + 1) = 22$. At the end, S_1 has 0 elements, and S_2 has 2.

(b) An insertion always takes 1 unit, so our worst-case cost must be caused by a removal. No more that n elements can ever be in S_1, and no fewer than 0 elements can be in S_2. Therefore the worst-case cost is $2n + 1$: $2n$ units to dump, and one extra to pop from S_2. This bound is tight, as seen by the following sequence: perform n insertions, then n removals. The first removal will cause a dump of n elements plus a pop, for $2n + 1$ work.

(c) The tightest amortized upper bounds are 3 units per insertion, and 1 unit per removal. We will prove this 2 ways (using the accounting and potential methods; the aggregate method seems too weak to employ elegantly in this case). (We would also accept valid proofs of 4 units per insertion and 0 per removal, although this answer is looser than the one we give here.)

Here is an analysis using the accounting method: with every insertion we pay \$3: \$1 is used to push onto S_1, and the remaining \$2 remain attached to the element just inserted. Therefore every element in S_1 has \$2 attached to it. With every removal we pay \$1, which will (eventually) be used to pop the desired element off of S_2. Before that, however, we may need to dump S_1 into S_2; this involves popping each element off of S_1 and pushing it onto S_2. We can pay for these pairs of operations with the $2 attached to each element in S_1.

Now we analyze the structure using the potential method: let $|S_1^i|$ denote the number of elements in S_1 after the ith operation. Then the potential function ϕ on our structure Q_i (the state of the queue after the ith operation) is defined to be $\phi(Q_i) = 2|S_1^i|$. Note that $|S_1^i| \geq 0$ at all times, so $\phi(Q_i) \geq 0$. Also, $|S_1^0| = 0$ initially, so $\phi(Q_0) = 0$ as desired.

Now we compute the amortized costs: for an insertion, we have $S_1^{i+1} = S_1^i + 1$, and the actual cost $c_i = 1$, so

$$\hat{c}_i = c_i + \phi(Q_{i+1}) - \phi(Q_i) = 1 + 2(S_1^i + 1) - 2(S_1^i) = 3.$$

For a removal, we have two cases. First, when there is no dump from S_1 to S_2, the actual cost is 1, and $S_1^{i+1} = S_1^i$. Therefore $\hat{c}_i = 1$. When there is a dump, the actual cost is $2|S_1^i| + 1$, and we have $S_1^{i+1} = 0$. Therefore we get

$$\hat{c}_i = (2|S_1^i| + 1) + 0 - 2|S_1^i| = 1$$

as desired.

Problem 7-4. Shortest-Path Special Cases

(a) We make the following observation about Dijkstra's algorithm in this case: if i is the value returned by the most recent DELETE-MIN, then the priority queue only contains keys $i, i+1, \ldots, i+C, \infty$. This is because each element in the queue has key at least i, and is either not a neighbor of any vertex that has been removed from the queue (in which case its key is still ∞), or it is a neighbor of a vertex that has been removed. Such a neighbor is within i of the source vertex, so the vertex in question would have key at most $i + C$. Therefore by keeping an array as our priority queue (with $CV = O(V)$ entries), we can implement DELETE-MIN in $O(1)$ time by straightforward search in the array, for a new total running time of $O(V + E)$.

We can also make a direct transformation to a BFS problem, in the following way: split each edge with weight $w > 0$ into w edges (by adding $w - 1$ nodes in between). Contract (i.e., merge) vertices connected by edges of weight 0. This transformation increases the size of the graph by a factor of at most C (a constant), so the number of nodes in the new graph is still $O(V)$, and the number of edges $O(E)$. Therefore we can run a breadth-first search in time $O(V + E)$.

(b) (Note the correction to the original problem set: the desired time is $O((V+E)\lg\lg u)$.) Note that the priorities in the queue are the lengths of paths, so they may be up to length uV. Use a van Emde Boas queue, with universe $\{0 \ldots uV\}$, in Dijkstra's algorithm. Beacuse $u > V$, the running time of a vEB operation is $O(\lg\lg uV) = O(\lg\lg u^2) = O(\lg\lg u)$., Instead of decreasing keys (which we don't know how to do for vEB queues), we simply remove the old key and insert the new one. This is done at most $|E|$ times, so by modifying the analysis of the algorithm, we get a $O((V + E) \lg \lg u)$ running time.

(c) Store a bit vector of length u, initially all zeros. To insert an element with key x, set bit x to 1 (and update any pointers to auxiliary data). Maintain an index to which key the last DELETE-MIN returned. The DELETE-MIN procedure works as follows: starting from the current index, find the smallest key that exists in the queue (i.e., the index of the first non-zero bit) and return its element. Update the index accordingly. The total time over a sequence of k operations is $O(u)$ to make at most one full pass over the bit vector, plus $O(k)$ to do the deletions, for $O(u + k)$ as desired.

(d) We can use the monotone priority queue exactly as described above in Dijkstra's algorithm. We perform $O(|V|)$ DELETE-MIN operations, so the running time becomes $O(|V| + |E| + u)$.