Definitions and Notation

Problem 1: Define the following words, phrases and symbols.

1. Set $A = \{x, y\}$, subset $B \subseteq A$, proper subset $B \subset A$, multiset $\{x, y, y\}$, power set $P(A)$, cardinality $|A|$, infinite set, natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$, integers $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$, empty set $\emptyset$, union $A \cup B$, intersection $A \cap B$, Cartesian product $A \times B$, complement $\bar{A}$, sequence $(x, y)$, $k$-tuple $(x_1,x_2,\ldots,x_k)$.

2. Function $f : D \to R$, domain $D$, range $R$, mapping $\to$, one-to-one, onto, bijection (one-to-one, onto).

3. Relation $R = \{(d_1,r_1),(d_2,r_2),\ldots,(d_i,r_i)\}$, reflexive $\forall x, xRx$, symmetric $\forall x, y, xRy \iff yRx$, transitive $\forall x, y, z, xRy \land yRz \Rightarrow xRz$, equivalence (reflexive, symmetric, transitive).

4. Graph $G = (V,E)$, degree, path, simple path, cycle, strongly connected.

5. Alphabet (input/output) $\Sigma = \{a,b,c\}$, symbols $a$, string $w = baac$, length $|w|$, empty string $\epsilon$, substring (consecutive) $baa$, concatenation $w||w$ or $ww$, lexicographic ordering $(\epsilon,0,1,00,01,10,11,000,\ldots)$, language $L = \{w_1,w_2,\ldots,w_\ell\}$.

6. Boolean logic $\{0,1\}$, NOT $\neg p$, AND $p \land q$, OR $p \lor q$, XOR $p \oplus q$, implication $p \Rightarrow q$, equality $p \Leftrightarrow q$, distributive law $p \land (q \lor r) \Leftrightarrow (p \land q) \lor (p \land r)$.

7. Theorem, lemma, corollary, proof, intuition, induction (assumes $P(n)$), strong induction (assumes $P(0),P(1),\ldots,P(n)$).

8. $(\ast)$ Machine, Automata, language accepted by a machine, language recognized by a machine.

Proof Techniques

Problem 2: Set-Theoretic Equivalence: Recall that in order to prove two sets $A, B$ are equivalent, one must show that $A \subseteq B$ and $B \subseteq A$. Prove De Morgan’s Law that $A \cap B = \bar{A} \cup \bar{B}$.

Problem 3: Proof by Contradiction: 1. If there are 6 people at a party shaking hands, then there must be at least two people who shook hands with the same number of other people.

2. Generalization (Problem 0.12 from Sipser’s Text): In any graph with at least 2 nodes there are two nodes of equal degree.

Problem 4: Proof by Induction (Base Case) (Induction Hypothesis) (Inductive Step): Problem 0.11 from Sipser’s Text.

Find the error in the following proof that all horses are the same color.

Claim: In any set of $h$ horses, all horses in the set are the same color.

Proof: By induction on $h$.

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1Sipser, pg 4. Zero can also be included in $\mathbb{N}$.

2Observe the anomaly that 11 preceeds 000; length takes precedence.
Basis: For $h = 1$. In any set containing just one horse, all horses clearly are the same color.

Inductive Step: For $k \geq 1$, assume that the claim is true for $h = k$ and prove that it is true for $h = k + 1$. Take any set $H$ of $k + 1$ horses, we will show that all horses in this set are the same color. Remove one horse from this set to obtain the set $H_1$ with just $k$ horses. By the induction hypothesis, all the horses in $H_1$ are the same color. Now replace the removed horse and remove a different one to obtain the set $H_2$. By the same argument, all the horses in $H_2$ are the same color. Therefore, all the horses in $H$ must be the same color and the proof is complete.

Problem 5: Proof by Induction (Base Case) (Induction Hypothesis) (Inductive Step): Now correctly prove the following statement: $\forall n \in \mathbb{N}, n^3 - n$ is divisible by 6.