Readings: Sipser, Chapter 8 (the whole chapter).

**Problem 1:** (Sipser Exercise 8.1) Show that for any function $f : N \to N$, where $f(n) \geq n$, the space complexity class $\text{SPACE}(f(n))$ is the same whether you define the class by using the single-tape TM model or the two tape read-only TM model.

**Problem 2:** (Sipser Problem 8.10) The Japanese game go-moku is played by two players, “X” and “O”, on a $19 \times 19$ grid. Players take turns placing markers, and the first player to achieve 5 of his/her markers consecutively in a row, column, or diagonal, is the winner. Consider this game generalized to an $n \times n$ board. Let

$$GM = \{\langle P \rangle \mid P \text{ is a position in generalized go-moku, where player “X” has a winning strategy} \}.$$  

By a position we mean a board with markers placed on it, such as may occur in the middle of a play of the game. Show that $GM \in \text{PSPACE}$.

**Problem 3:** The proof of Savitch’s theorem, in Section 8.1, describes in general how one can simulate any $f(n)$-space-bounded nondeterministic Turing machine $N$ with an $f^2(n)$-space-bounded deterministic Turing machine $M$. The key is a recursive computation of the CANYIELD relation, which reuses space.

Give a good upper bound on the running time of $M$ on input $w$.

**Problem 4:** (Sipser Problem 8.12) Show that TQBF restricted to formulas where the part following the quantifiers is in conjunctive normal form is still PSPACE-complete.

**Problem 5:** (Sipser Problem 8.17) Let $A$ be the language of properly nested parentheses. For example, $()$ and $((()))$ are in $A$, but $)$ is not. Show that $A$ is in $L$.

**Problem 6:** Show:
1. $A \leq_L B \Rightarrow \bar{A} \leq_L \bar{B}$.
2. $A \leq_L B$ and $B \in \text{NL} \Rightarrow A \in \text{NL}$.
3. $A \leq_L B$ and $B \leq_L C \Rightarrow A \leq_L C$.

**Problem 7:** (Sipser 8.27) Recall that a directed graph is strongly connected if every two nodes are connected by a directed path in each direction. Let

$$\text{STRONGLY-CONNECTED} = \{\langle G \rangle \mid G \text{ is a strongly connected graph}\}.$$  

Show that STRONGLY-CONNECTED is NL-complete.

**Problem 8:** This problem uses the ideas in the proof of Theorem 8.27.

Describe a nondeterministic log-space Turing machine $M$ that decides the language

$$L = \{\langle G, s, m, k \rangle \mid G \text{ is a directed graph, } s \text{ is a node in } G, m, k \in \mathbb{N}, \text{ and exactly}

m \text{ nodes of } G \text{ are reachable from } s \in G \text{ by paths consisting of at most } k \text{ edges}\}.$$  

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That is, if exactly $m$ nodes are reachable from $s \in G$ by paths of length at most $k$, than $M$ must accept $\langle G, s, m, k \rangle$ on some computation path. On the other hand, if more or fewer than $m$ nodes are reachable from $s \in G$ by paths of length at most $k$, then $M$ must reject $\langle G, s, m, k \rangle$ on all computation paths.

Explain why $M$ works correctly and why it works in log space.