6.045J/18.400J: Automata, Computability and Complexity Nancy Lynch

Homework 5: Solutions

Due: March 16, 2005 *Vinod Vaikuntanathan*

Problem 1: Prove that the following languages are undecidable. Use reductions from A_{TM} or other problems already known to be undecidable. In all these problems, $\Sigma = \{0, 1\}$.

1. $L_1 = \{< M >: M$ is a Turing machine and M accepts the string $001\}$.
2. $L_2 = \{< M >: M$ is a Turing machine, M accepts 001 and M does not accept $110\}$.
3. $L_3 = \{< M >: M$ is a Turing machine and M accepts exactly the strings that are palindromes$\}$.

Solution 1:

1.1: We will reduce from A_{TM} to prove that L_1 is undecidable. Let D be a TM that decides L_1. We could then construct a decider S for A_{TM} as follows.

S="On input $< M, w >$, an encoding of a TM M and a string w,

1. Construct TM R from M and w as detailed below.
2. Run D on $< R >$.
3. If D accepts, accept; otherwise, reject."

R="On input x,

1. If $x \neq 001$, reject.
2. Otherwise, run M on w.
3. If M accepts, accept; otherwise, reject."

Thus, we contrive that R will only accept 001 if and only if M accepts w. Notice that R did not need to reject all strings not equal to 001 (it could have accepted them).

1.2: (Note that this is the same construction from part 1)
We will reduce from A_{TM} to prove that L_2 is undecidable. Let D be a TM that decides L_2. We could then construct a decider S for A_{TM} as follows.

S="On input $< M, w >$, an encoding of a TM M and a string w,

1. Construct TM R from M as detailed below.
2. Run D on $< R >$.
3. If D accepts, accept; otherwise, reject."

R="On input x,

1. If $x \neq 001$, reject.

2. Else, Run M on w.

3. If M accepts, accept.

4. If M rejects, reject.

If M accepts w, then $110 \notin L(R)$ and $001 \in L(R)$. Therefore, $R \in L_2$. If M does not accept w, $001 \notin L(R)$. Therefore, $R \notin L_2$. Thus, $R \in L_2$ if and only if $< M, w > \in A_{TM}$.

1.3: We will reduce from A_{TM} to prove that L_3 is undecidable. Let D be a TM that decides L_3. We could then construct a decider S for A_{TM} as follows.

S="On input $< M, w >$, an encoding of a TM M and a string w,

1. Construct TM R from M and w as detailed below.

2. Run D on $< R >$.

3. If D accepts, accept; otherwise, reject."

R="On input x,

1. Check if x is a palindrome. If it is not, reject.

2. Otherwise, run M on w.

3. If M accepts, accept; otherwise, reject."

Thus, when M accepts w, R accepts exactly the strings that are palindromes. Therefore, in this case, $R \in L_3$. When M does not accept w, R does not accept any string, and therefore $R \notin L_3$. $< M, w > \in A_{TM}$ if and only if $R \in L_3$.

Note 1: Notice that these reductions are indeed mapping reductions. Also observe that, the machine S is the same in all the cases and it is only the description of R that changes (if at all).

Problem 2: Let $A = \{< D_1 >, < D_2 >, < D_3 >, \ldots\}$ be a language consisting of string representations of Turing machines that are deciders, that is, each machine D_i halts (accepts or rejects) on every input. *The goal of this problem is to prove that A is not Turing-recognizable.*

Suppose, for contradiction, that A is Turing-recognizable, and therefore, enumerable by an enumerator machine E. Show that A cannot include deciders for all decidable languages—some decidable language must be left out. That is, there is some decidable language L that is not equal to $L(D_i)$ for any i.

(Hint: Recall the diagonalization method; try constructing L using the enumerator E.)

Solution 2: Suppose that $L = \{< D_1 >, < D_2 >, < D_3 >, \ldots\}$ is Turing-recognizable and let E be the enumerator that enumerates it. We construct a decidable language $D' \notin L$ as follows.

D'="On input w,

1. Run E until $< D_w >$, the wth TM description, is printed.

2. Run D_w on w.

3. If D accepts, reject; otherwise, accept."

We need to show two things.

Claim 1: D' is a decider.
Proof: Since w must be a finite string, we know that E will print $< D_w >$ in finite time. Since D_w is a decider, we know that it will halt when run on w. Thus, D' will always be able to halt (i.e., accept or reject) on any input w as well. ■

Claim 2: D' is not in L.
Proof: We secured this property by using the diagonalization method in our construction of D'. Suppose D' were in L, meaning $D' = D_i \in L$. Then on input i, D_i would either accept or reject, and D' would *do the opposite* of D_i. Thus, $L(D') \neq L(D_i)$ for all $D_i \in L$. ■

Problem 3: Find a match in the following instance of the PCP:

$$\left\{ \left[\frac{ab}{abab}\right], \left[\frac{b}{a}\right], \left[\frac{aba}{b}\right], \left[\frac{aa}{a}\right] \right\}$$

Solution 3: You must indicate which dominos are used, in which order to receive credit.

$$\left[\frac{ab}{abab}\right] \left[\frac{ab}{abab}\right] \left[\frac{aba}{b}\right] \left[\frac{b}{a}\right] \left[\frac{b}{a}\right] \left[\frac{aa}{a}\right] \left[\frac{aa}{a}\right]$$

A better solution, that some of you found, is the following:

$$\left[\frac{aa}{a}\right] \left[\frac{aa}{a}\right] \left[\frac{b}{a}\right] \left[\frac{ab}{abab}\right]$$

Problem 4: Consider the machine M_1 on page 132 of (the old edition of) Sipser's book, which recognizes the language consisting of all strings of the form $w\#w$, where $w \in \{0,1\}^*$.

1. Write out the accepting computation history for the machine M_2 on input $0\#0$.
2. What are the tiles for the instance of the Modified Post Correspondence Problem defined for M_1 and input $0\#0$? What is the initial tile?
3. Write out your computation history from part (a) twice, one copy above the other. Draw lines indicating how your tiles from part (b) can be used to match these two copies.

Solution 4:

4.1: The accepting computation history for M_2 on 00 is:

1. $q_1 0\#0$
2. $\sqcup q_2 \#0$
3. $\sqcup\# q_4 0$
4. $\sqcup\#0 q_4 \sqcup$
5. $\sqcup\# q_6 0\sqcup$
6. $\sqcup q_6 \#0\sqcup$

7. $q_6 \sqcup \#0\sqcup$

8. $\sqcup q_8 \#0\sqcup$

9. $\sqcup \# q_{10} 0\sqcup$

10. $\sqcup q_{12} \# x\sqcup$

11. $q_{12} \sqcup \# x\sqcup$

12. $\sqcup q_{13} \# x\sqcup$

13. $\sqcup \# q_{14} x\sqcup$

14. $\sqcup \# x q_{14}\sqcup$

15. $\sqcup \# x \sqcup q_{accept}$

Note that we could have drawn any number of $\sqcup$ symbols to the far right of the tape, but starting in configuration 4, we must draw at least one, since we must indicate to which symbol the tape head points. (It is okay to leave this off for the last configuration, since the machine halts.)

4.2: See pages 184-189 of (the old edition of) the text. We will give the dominos for the MPCP. Here, $\Gamma = \{0, \#, x, \sqcup\}$ and $Q = \{q_1, q_2, q_4, q_6, q_8, q_{10}, q_{12}, q_{13}, q_{14}, q_{accept}\}$. Note that we have included for the domino construction only those states that are ever visited in the computation, and those transitions in the left branch of the TM computation tree (see the text). One could construct the full set of dominoes, but they will never be used.

Part 1:

$$\left[\frac{\natural}{\natural q_1 0\#0\natural}\right]$$

Part 2:

$$\left[\frac{q_1 0}{\sqcup q_2}\right], \left[\frac{q_2 0}{0 q_2}\right], \left[\frac{q_2 1}{1 q_2}\right], \left[\frac{q_2 \#}{\# q_4}\right], \left[\frac{q_4 0}{0 q_4}\right], \left[\frac{q_4 1}{1 q_4}\right], \left[\frac{q_6 \sqcup}{\sqcup q_6}\right], \left[\frac{q_8 0}{0 q_8}\right],$$

$$\left[\frac{q_8 1}{1 q_8}\right], \left[\frac{q_8 \#}{\# q_{10}}\right], \left[\frac{q_{10} x}{x q_{10}}\right], \left[\frac{q_{12} \sqcup}{\sqcup q_{13}}\right], \left[\frac{q_{13} x}{x q_{13}}\right], \left[\frac{q_{13} \#}{\# q_{14}}\right], \left[\frac{q_{13} 0}{x q_8}\right], \left[\frac{q_{14} x}{x q_{14}}\right], \left[\frac{q_{14} \sqcup}{\sqcup q_{accept}}\right]$$

Part 3:

$$\left[\frac{0 q_4 \sqcup}{q_6 0 x}\right], \left[\frac{x q_4 \sqcup}{q_6 x x}\right], \left[\frac{\# q_4 \sqcup}{q_6 \# x}\right], \left[\frac{\sqcup q_4 \sqcup}{q_6 \sqcup x}\right],$$

$$\left[\frac{0 q_6 0}{q_6 00}\right], \left[\frac{x q_6 0}{q_6 x 0}\right], \left[\frac{\sqcup q_6 0}{q_6 \sqcup 0}\right], \left[\frac{\# q_6 0}{q_6 \# 0}\right],$$

$$\left[\frac{0 q_6 1}{q_6 01}\right], \left[\frac{x q_6 1}{q_6 x 1}\right], \left[\frac{\sqcup q_6 1}{q_6 \sqcup 1}\right], \left[\frac{\# q_6 1}{q_6 \# 1}\right],$$

$$\left[\frac{0 q_6 \#}{q_6 0 \#}\right], \left[\frac{x q_6 \#}{q_6 x \#}\right], \left[\frac{\sqcup q_6 \#}{q_6 \sqcup \#}\right], \left[\frac{\# q_6 \#}{q_6 \#\#}\right],$$

$$\left[\frac{0 q_{10} 0}{q_{12} 0 x}\right], \left[\frac{x q_{10} 0}{q_{12} x x}\right], \left[\frac{\sqcup q_{10} 0}{q_{12} \sqcup x}\right], \left[\frac{\# q_{10} 0}{q_{12} \# x}\right],$$

$$\left[\frac{0 q_{12} 0}{q_{12} 00}\right], \left[\frac{x q_{12} 0}{q_{12} x 0}\right], \left[\frac{\sqcup q_{12} 0}{q_{12} \sqcup 0}\right], \left[\frac{\# q_{12} 0}{q_{12} \# 0}\right],$$

$$\left[\frac{0q_{12}x}{q_{12}0x}\right], \left[\frac{xq_{12}x}{q_{12}xx}\right], \left[\frac{\sqcup q_{12}x}{q_{12}\sqcup x}\right], \left[\frac{\#q_{12}x}{q_{12}\#x}\right],$$

$$\left[\frac{0q_{12}\sqcup}{q_{12}0\sqcup}\right], \left[\frac{xq_{12}\sqcup}{q_{12}x\sqcup}\right], \left[\frac{\sqcup q_{12}\sqcup}{q_{12}\sqcup\sqcup}\right], \left[\frac{\#q_{12}\sqcup}{q_{12}\#\sqcup}\right],$$

$$\left[\frac{0q_{12}\#}{q_{12}0\#}\right], \left[\frac{xq_{12}\#}{q_{12}x\#}\right], \left[\frac{\sqcup q_{12}\#}{q_{12}\sqcup\#}\right], \left[\frac{\#q_{12}\#}{q_{12}\#\#}\right]$$

Part 4:

$$\left[\frac{0}{0}\right], \left[\frac{x}{x}\right], \left[\frac{\sqcup}{\sqcup}\right], \left[\frac{\#}{\#}\right]$$

Part 5:

$$\left[\frac{\natural}{\natural}\right], \left[\frac{\natural}{\sqcup\natural}\right]$$

Part 6:

$$\left[\frac{0q_{accept}}{q_{accept}}\right], \left[\frac{xq_{accept}}{q_{accept}}\right], \left[\frac{\sqcup q_{accept}}{q_{accept}}\right], \left[\frac{\#q_{accept}}{q_{accept}}\right], \left[\frac{q_{accept}0}{q_{accept}}\right], \left[\frac{q_{accept}x}{q_{accept}}\right], \left[\frac{q_{accept}\#}{q_{accept}}\right], \left[\frac{q_{accept}\sqcup}{q_{accept}}\right]$$

Part 7:

$$\left[\frac{q_{accept}\natural\natural}{\natural}\right]$$

4.3: Here is the match in the MPCP problem that shows M has an accepting computation history on w.

$$\left[\frac{\natural}{\natural q_1 0\#0\natural}\right] \left[\frac{q_1 0}{\sqcup q_2}\right] \left[\frac{\#}{\#}\right] \left[\frac{0}{0}\right] \left[\frac{\natural}{\natural}\right]$$
$$\left[\frac{\sqcup}{\sqcup}\right] \left[\frac{q_2\#}{\#q_4}\right] \left[\frac{0}{0}\right] \left[\frac{\natural}{\natural}\right]$$
$$\left[\frac{\sqcup}{\sqcup}\right] \left[\frac{\#}{\#}\right] \left[\frac{q_4 0}{0q_4}\right] \left[\frac{\natural}{\sqcup\natural}\right]$$
$$\left[\frac{\sqcup}{\sqcup}\right] \left[\frac{\#}{\#}\right] \left[\frac{0q_4\sqcup}{q_6 0\sqcup}\right] \left[\frac{\natural}{\natural}\right]$$
$$\left[\frac{\sqcup}{\sqcup}\right] \left[\frac{\#q_6 0}{q_6\#0}\right] \left[\frac{\sqcup}{\sqcup}\right] \left[\frac{\natural}{\natural}\right]$$
$$\left[\frac{\sqcup q_6\#}{q_6\sqcup\#}\right] \left[\frac{0}{0}\right] \left[\frac{\sqcup}{\sqcup}\right] \left[\frac{\natural}{\natural}\right]$$
$$\left[\frac{q_6\sqcup}{\sqcup q_8}\right] \left[\frac{\#}{\#}\right] \left[\frac{0}{0}\right] \left[\frac{\sqcup}{\sqcup}\right] \left[\frac{\natural}{\natural}\right]$$
$$\left[\frac{\sqcup}{\sqcup}\right] \left[\frac{q_8\#}{\#q_{10}}\right] \left[\frac{0}{0}\right] \left[\frac{\sqcup}{\sqcup}\right] \left[\frac{\natural}{\natural}\right]$$
$$\left[\frac{\sqcup}{\sqcup}\right] \left[\frac{\#q_{10}0}{q_{12}\#x}\right] \left[\frac{\sqcup}{\sqcup}\right] \left[\frac{\natural}{\natural}\right]$$

$$\left[\frac{\sqcup q_{12}\#}{q_{12}\sqcup\#}\right]\left[\frac{x}{x}\right]\left[\frac{\sqcup}{\sqcup}\right]\left[\frac{\sharp}{\sharp}\right]$$

$$\left[\frac{q_{12}\sqcup}{\sqcup q_{13}}\right]\left[\frac{\#}{\#}\right]\left[\frac{x}{x}\right]\left[\frac{\sqcup}{\sqcup}\right]\left[\frac{\sharp}{\sharp}\right]$$

$$\left[\frac{\sqcup}{\sqcup}\right]\left[\frac{q_{13}\#}{\# q_{14}}\right]\left[\frac{x}{x}\right]\left[\frac{\sqcup}{\sqcup}\right]\left[\frac{\sharp}{\sharp}\right]$$

$$\left[\frac{\sqcup}{\sqcup}\right]\left[\frac{\#}{\#}\right]\left[\frac{q_{14}x}{x q_{14}}\right]\left[\frac{\sqcup}{\sqcup}\right]\left[\frac{\sharp}{\sharp}\right]$$

$$\left[\frac{\sqcup}{\sqcup}\right]\left[\frac{\#}{\#}\right]\left[\frac{x}{x}\right]\left[\frac{q_{14}\sqcup}{\sqcup q_{accept}}\right]\left[\frac{\sharp}{\sharp}\right]$$

Cleaning up,

$$\left[\frac{\sqcup}{\sqcup}\right]\left[\frac{\#}{\#}\right]\left[\frac{x}{x}\right]\left[\frac{\sqcup q_{accept}}{q_{accept}}\right]\left[\frac{\sharp}{\sharp}\right]$$

$$\left[\frac{\sqcup}{\sqcup}\right]\left[\frac{\#}{\#}\right]\left[\frac{x q_{accept}}{q_{accept}}\right]\left[\frac{\sharp}{\sharp}\right]$$

$$\left[\frac{\sqcup}{\sqcup}\right]\left[\frac{\# q_{accept}}{q_{accept}}\right]\left[\frac{\sharp}{\sharp}\right]$$

$$\left[\frac{\sqcup q_{accept}}{q_{accept}}\right]\left[\frac{\sharp}{\sharp}\right]$$

$$\left[\frac{q_{accept}\sharp\sharp}{\sharp}\right]$$

Sigh !! That _was_ tedious.