6.045J/18.400J: Automata, Computability and Complexity	Prof. Nancy Lynch
Homework 1	
Due: February 9, 2005	*Vinod Vaikuntanathan*

Note: Recitation meets on Thursday, Feb 3 at 10am in 34-302 and 1pm and 4pm in 34-304.

Problem 1: Let $\mathbf{Z}$ be the set of Integers. Let R be the binary relation on $\mathbf{Z}$ such that $\{a, b\} \in R$ if and only if $ab \geq 0$.

(a) Is R reflexive? Explain.

(b) Is R symmetric? Explain.

(c) Is R transitive? Explain.

(d) Define a relation $R_1 \subseteq R$ that is reflexive and symmetric but not transitive.

(e) Define a relation $R_2 \subseteq R$ that is reflexive and transitive but not symmetric.

(f) Define a relation $R_3 \subseteq R$ that is symmetric and transitive but not reflexive.

(g) Define a relation $R_4 \subseteq R$ that is an equivalence relation.

Solution 1:

(a) R is reflexive. Recall that, for every $a \in \mathbf{Z}$, $a^2 \geq 0$. This means that, for every $a \in \mathbf{Z}$, $\{a, a\} \in R$. Therefore R is reflexive.

(b) R is symmetric. Suppose $\{a, b\} \in R$. This means $ab \geq 0$. Since multiplication is commutative in $\mathbf{Z}$, $ba = ab \geq 0$ too. Therefore, $\{b, a\} \in R$. We just proved that, if $\{a, b\} \in R$, $\{b, a\} \in R$ too. Thus R is symmetric.

(c) R is *not* transitive. As a counterexample, observe that $\{-1, 0\} \in R$ and $\{0, 1\} \in R$, but $\{-1, 1\}$ is not in R.

There are many correct answers to the following parts of the problem. Here is one example for each part.

(d) $R_1 = R$. It is reflexive and symmetric, but not transitive.

(e) R_2 is defined as: $\{a, b\} \in R_2$ if and only if either $a \geq b \geq 0$ or $a = b$. Suppose $\{a, b\} \in R_2$. Then, either $a \geq b \geq 0$, in which case $ab \geq 0$ (or) $a = b$, in which case $ab = a^2 \geq 0$. Thus, if $\{a, b\} \in R_2$, $\{a, b\} \in R$. Therefore, $R_2 \subseteq R$. R_2 is reflexive, because $\{a, a\} \in R_2$ for every $a \in \mathbf{Z}$ (by definition of R_2). R_2 is not symmetric, because $\{2, 1\} \in R_2$, but $\{1, 2\}$ is not. To prove that R_2 is transitive, observe that $\{a, b\} \in R_2$ implies that either $a \geq b \geq 0$ or $a = b$. $\{b, c\} \in R_2$ implies $b \geq c \geq 0$ or $b = c$. Consider the following four cases:

- $a \geq b \geq 0$ and $b \geq c \geq 0$. Clearly, $a \geq b \geq c \geq 0$. Therefore, $a \geq c \geq 0$, and $\{a, c\} \in R_2$.
- $a \geq b \geq 0$ and $b = c$. Here, again $a \geq c \geq 0$ and $\{a, c\} \in R_2$.
- $a = b$ and $b \geq c \geq 0$. Clearly, again $a \geq c \geq 0$ and $\{a, c\} \in R_2$.
- $a = b$ and $b = c$. $a = c$. Therefore, $\{a, c\} \in R_2$.

Thus, whenever $\{a, b\} \in R_2$ and $\{b, c\} \in R_2$, $\{a, c\} \in R_2$. R_2 is transitive.

(f) Define R_3 to be the empty set. R_3 is clearly a subset of R. Additionally, the empty relation is symmetric and transitive, but not reflexive.

(g) Define R_4 as the set of all $\{a, b\}$ such that either $ab > 0$ or $a = b = 0$. This is clearly a subset of R. R_4 is reflexive because for all $a \in \mathbf{Z}$ such that $a \neq 0$, $a^2 > 0$, and additionally, $\{0, 0\} \in R_4$. R_4 is symmetric (again, this should be clear). To prove that R_4 is transitive, assume that $\{a, b\} \in R_4$ and $\{b, c\} \in R_4$. Therefore, either $a = b = c = 0$, in which case $\{a, c\} \in R_4$, or $ab > 0$ and $bc > 0$, in which case, $ab^2c > 0$. Since $b^2 > 0$, $ac > 0$ too. Therefore, $\{a, c\} \in R_4$.

Problem 2: Construct truth tables for each of the following formulas. Also, for each pair of formulas, state which of the following holds:

- They are equivalent,
- They are not equivalent, but one implies the other (say which is which), or
- Neither of the above.

(a) $p \oplus (q \Rightarrow \neg p)$

(b) $(q \Rightarrow \neg p) \Rightarrow \neg p$

(c) $(p \Rightarrow q) \Rightarrow \neg p$

(d) $p \wedge \neg q \wedge (p \Rightarrow q)$

Solution 2:

(a)

p	q	$\neg p$	$q \Rightarrow \neg p$	$(p \oplus (q \Rightarrow \neg p)$
f	f	t	t	t
f	t	t	t	t
t	f	f	t	f
t	t	f	f	t

(b)

p	q	$\neg p$	$q \Rightarrow \neg p$	$(q \Rightarrow \neg p) \Rightarrow \neg p$
f	f	t	t	t
f	t	t	t	t
t	f	f	t	f
t	t	f	f	t

(c)

p	q	$\neg p$	$p \Rightarrow q$	$(p \Rightarrow q) \Rightarrow \neg p$
f	f	t	t	t
f	t	t	t	t
t	f	f	f	t
t	t	f	t	f

(d)

p	q	$\neg q$	$p \wedge \neg q$	$p \Rightarrow q$	$p \wedge \neg q \wedge (p \Rightarrow q)$
f	f	t	f	t	f
f	t	f	f	t	f
t	f	t	t	f	f
t	t	f	f	t	f

We see that (a) $\Leftrightarrow$ (b), since their truth tables are identical. Further, (d) implies both (a=b) and (c), since a false statement implies anything (that's why they won't be convincing in a homework proof!), but neither (a=b) nor (c) implies (d), since true never implies false. Finally, we claim that neither (a=b) $\Rightarrow$ (c) nor (c) $\Rightarrow$ (a=b), because each formula would also require that true implies false.

Problem 3:

(a) Prove that for every natural number $n \geq 12$, there exist integers a, b such that $n = 3a + 7b$.

(b) A *Hamiltonian cycle* in an undirected graph is a cycle that goes through every vertex in the graph exactly once.
Suppose that G is an undirected graph that has a Hamiltonian cycle. Suppose that H is another undirected graph that is obtained from G by adding one node at a time, along with some edges between the new node and some of the old nodes.
More precisely, we have a sequence of graphs $G = G_0, G_1, G_2, \ldots, G_k = H$, where each graph G_{i+1} is obtained from the previous graph G_i by adding one node n_{i+1}, together with edges connecting the new node n_{i+1} to strictly more than half of the nodes in the previous graph G_i. Prove that H also must have a Hamiltonian cycle.

Solution 3:

(a) Proof by Induction.

- Base case: $n = 12$. Here we see that $12 = 3(4) + 7(0)$.
- Inductive Hypothesis: Assume there is some natural number $n \geq 12$ and integers a, b such that $n = 3a + 7b$.
- Inductive Step: Given the inductive hypothesis, we wish to show that there exist integers c, d such that $n + 1 = 3c + 7d$. Recall that $3 * 2 = 6$ and $7 * 1 = 7$. Thus, to "add one" to n, we simply "add 7" and "subtract 6" as follows: $n + 1 = 3a + 7b - 3(2) + 7(1) = 3(a-2) + 7(b+1)$. Thus, we see that integers $c = a - 2$ and $d = b + 1$ do, in fact, exist.

(b) Proof by Induction.

- Base Case: $i = 0$. Immediate from the problem statement (It is given that $G = G_0$ has a Hamiltonian cycle).
- Inductive Hypothesis: Assume that G_i has a Hamiltonian cycle.
- Inductive Step: We want to show that G_{i+1} has a Hamiltonian cycle, given the inductive hypothesis. Consider the set of all vertices in G_i that are adjacent to n_{i+1}. We claim that there exist two of those vertices (say u and v) that are adjacent in the Hamiltonian cycle. The new Hamiltonian cycle in G_{i+1} can be constructed by removing the edge between u and v and adding two new edges – $\{u, n_{i+1}\}$ and $\{n_{i+1}, v\}$.